Some Basic Principles of Combinatorics

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Combinatorics gives us tools to count these large amounts in a short time.

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Sets with structure

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We use the special symbol $\mathbb{N}$ for the "counting numbers" (officially the *natural numbers*). That is $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. This is an infinite set.

Two examples

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The set of 5-letter strings (lets say they contain only uppercase English letters) can be generated by an algorithm using loops of length 26 nested 5 levels deep. A loop of length 26 inside a loop of length 26 has its contents executed 26×26 times. Extending this to 5 levels deep gives us $26 \times 26 \times 26 \times 26 \times 26$, which is the number 11,881,376 we saw before.

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If we put these together we get: if B is in 1-to-1 correspondence with a subset of A then $|B| \leq |A|$.

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Example: $|\{1,2,3\}| = |\{1,2\}| + |\{2,3\}| - |\{2\}|$.

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The rule of sum extends to any number of tasks, as long as no two tasks can be performed simultaneously.

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1. The number of ways to perform a task does not depend on the outcome of the previous tasks.
2. If we use this to count final outcomes, we must have
   - If any one task is done differently, the final outcome is different.
   - All final outcomes are produced by some way of performing the sequence of tasks.

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There are 26 ways to perform step 1, 25 ways to perform step 2, etc. As before, a different choice at any step produces a different string. Every string is the result of some sequence of choices. So there are $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = 7{,}893{,}600$ possibilities.

A picture

The following illustrates building strings of length 2 from the letters A, B, and C without repeated letters. This illustrates the rule of product: the first task (select a letter) produces a 3-way branching, and then the second task (select a different letter) produces 2-way branching. There are a total of $3 \cdot 2$ paths to the different final results.