Review of Chapter 6

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In fact, this case is the only way a matrix can be diagonalizable:

Theorem

An $n \times n$ matrix A is diagonalizable if and only if $\mathbb{R}^n$ has a basis consisting of eigenvectors of A. It happens when the sum of the dimensions of the eigenspaces is n. In that case the diagonal elements are the eigenvalues of A.

Example: Diagonalize the following matrix, or else prove it is not diagonalizable:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

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First step: find the eigenvalues.

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (2-\lambda)((1-\lambda)^2 - 1) = (2-\lambda)(\lambda^2 - 2\lambda)$$

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Equating this to 0 we get two roots: $\lambda = 0$ and a double root $\lambda = 2$.

Second step: Find the eigenspaces.

$$A - 2I = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \xrightarrow{\text{5 EROs}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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The equations this gives are $x_1 = x_3$ and $x_2 = 0$. The eigenspace is therefore

$$\left\{ \begin{pmatrix} \alpha \\ 0 \\ \alpha \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\} \quad \text{with basis} \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

The eigenspace for $\lambda = 0$

$$A - 0I = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{3 EROs}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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There is one special case where a matrix is assured to be diagonalizable: when there are as many different eigenvalues as the dimension.

This is because each eigenspace has dimension at least one, and in this case there will be n eigenspaces.

Some Examples

(1) Find the eigenvalues of the matrix $A = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 1 \\ 2 & -1 & 1 \end{pmatrix}$

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Find $\det(A - \lambda I)$

$$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 1 & 3-\lambda & 1 \\ 2 & -1 & 1-\lambda \end{vmatrix} = (3-\lambda)(\lambda^2 - 4\lambda + 4)$$

Equate this to 0 and solve to get $\lambda = 3, 2, 2$.

(2) For each eigenvalue of A, find a basis for its eigenspace.

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For $\lambda = 3$, the matrix

$$A - 3I = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & -1 & -2 \end{pmatrix} \quad \text{reduces to} \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

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So solutions of $(A - 3I)\mathbf{x} = 0$ are $\mathbf{x} = \begin{pmatrix} -\alpha \\ -4\alpha \\ \alpha \end{pmatrix}$,

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(3) We can conclude that A is not diagonalizable.

(1) Find the eigenvalues of the matrix $B = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & -2 & 0 \end{pmatrix}$

The determinant of $B - \lambda I$ is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 3-\lambda & 1 \\ 0 & -2 & -\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 3\lambda + 2)$$

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For $\lambda = 1$ the matrix

$$B - I = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{pmatrix} \quad \text{reduces to} \quad \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

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(3) We can conclude that B is diagonalizable. The invertible matrix $S = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 1 & 2 \end{pmatrix}$, consisting of the eigenvectors of B, will satisfy $S^{-1}BS = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

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Theorem

Let A be a symmetric $n \times n$ matrix, (i.e., $A^T = A$) and suppose λ_1 and λ_2 are two different eigenvalues. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are the respective eigenvectors then $\mathbf{x}_1 \perp \mathbf{x}_2$.

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Theorem

If A is a symmetric matrix then all eigenvalues are real, and there an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

Example: Find an orthonormal basis of eigenvectors for $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

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Eigenvalues: $\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)((2-\lambda)^2 - 1) = (2-\lambda)(\lambda^2 - 4\lambda + 3)$.

Equate to zero to get $\lambda = 1, 2, 3$.

For $\lambda = 1$, the matrix

$$A - I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

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A basis for the eigenspace is $\begin{pmatrix} 0 & -1 & 1 \end{pmatrix}^T$. Normalize to get $\mathbf{q}_1 = \begin{pmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}^T$.

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For $\lambda = 2$ the matrix

$$A - 2I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

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A basis for the eigenspace is $\mathbf{q}_2 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$. The norm is already 1. Note that $\mathbf{q}_1 \perp \mathbf{q}_2$.

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For example: let $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ For this matrix $\det(A - \lambda I) = -(\lambda - 1)^2(\lambda - 4)$, giving eigenvalues $\lambda = 1, 1, 4$.

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For $\lambda = 1$, the matrix $A - I$ is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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$$\left(\begin{array}{ccc} -1 & 1 & 0 \end{array} \right)^T \text{ and } \left(\begin{array}{ccc} -1 & 0 & 1 \end{array} \right)^T$$

These are not orthogonal, though they are a basis for the eigenspace. If we want an orthonormal basis we can apply the Gram-Schmidt process to these two to get

$$\left(\begin{array}{ccc} -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{array} \right) \text{ and } \left(\begin{array}{ccc} -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{array} \right)^T$$

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$$\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{pmatrix}^T$$

These two are orthogonal to the eigenvector for $\lambda = 4$. If we normalize that we get $\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$

The last matrix corresponds to the equation $x_1 + x_2 + x_3 = 0$. Setting $\mathbf{x}_2 = \alpha$ and $x_3 = \beta$ we get solutions $\mathbf{x} = \begin{pmatrix} -\alpha - \beta & \alpha & \beta \end{pmatrix}^T$. And the usual way to obtain a basis gives us

$$\begin{pmatrix} -1 & 1 & 0 \end{pmatrix}^T \text{ and } \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}^T$$

These are not orthogonal, though they are a basis for the eigenspace. If we want an orthonormal basis we can apply the Gram-Schmidt process to these two to get

$$\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{pmatrix}^T$$

These two are orthogonal to the eigenvector for $\lambda = 4$. If we normalize that we get $\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$ and the three together give us an orthonormal basis for $\mathbb{R}^3$ of eigenvectors for A.