Review of Chapter 5, part 2, plus Chapter 6

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Theorem (Gram-Schmidt)

If $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ *is an independent set in an inner product space* V *then there exists and orthonormal set* $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ *such that*
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When computing by hand, it is quite a bit easier to first create an orthogonal (but not orthonormal) set $\mathbf{v}_1, \ldots, \mathbf{v}_r$ and then normalize them at the end by $\mathbf{u}_j = (1/\|\mathbf{v}_j\|)\mathbf{v}_j$.

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- Steps $k > 3$: set $\mathbf{v}_k = \mathbf{x}_k - \mathbf{p}_{k-1}$ where $\mathbf{p}_{k-1}$ is the projection of $\mathbf{x}_k$ onto $\mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_{k-1})$.

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It is permissible to replace any $\mathbf{v}_k$ by any nonzero multiple of itself before going on to finding $\mathbf{v}_{k+1}$. That can make the later steps a bit easier.

Example: find an orthonormal basis for the span of

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \ \mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{x}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

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Definition (Eigenvalue/eigenvector)

If A is an $n \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$ is a **nonzero** vector such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ then $\mathbf{x}$ is called an *eigenvector* for A and λ is an *eigenvalue*. For a given eigenvalue λ, the set of solutions of $A\mathbf{x} = \lambda\mathbf{x}$ is called the *eigenspace associated to λ*.

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We find eigenvectors by finding the nontrivial solutions of $(A - \lambda I)\mathbf{x} = \mathbf{0}$ once we know the eigemvalues

Ideally, for applications, we want a basis of eigenvectors. If we have that, say $\mathcal{B} = [\mathbf{v}_1, \mathbf{v}_1, \ldots, \mathbf{v}_n]$ is a basis and $A\mathbf{v}_j = \lambda_j \mathbf{v}_j$ for every j, then we have the following:

Suppose $[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$. This means that $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$

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Moreover, if $S = \begin{pmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_n \end{pmatrix}$ (the transition matrix from $\mathcal{B}$ to $\mathcal{E}$), then $S^{-1}AS$ is a diagonal matrix, with eigenvalues of A on the diagonal.

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Find the eigenvalues of $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$.

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Find the eigenspaces for these eigenvalues.

The nullspace of $A - 2I$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{\text{2 EROs}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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So we get a basis for this eigenspace: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

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Then x_1 and x_2 are the leading variables and x_3 is free. The equations are $x_1 = 0$ and $x_2 = -x_3$. This gives the eigenspace

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Because we have altogether 3 independent vectors, we have a basis for $\mathbb{R}^3$ consisting of eigenvectors. If we put the basis vectors in a matrix S and find its inverse S^{-1}, then the product $S^{-1}AS$ will be a diagonal matrix D with $2, 2, 1$ on the diagonal:

That is, $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$, $S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

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Note that $A = SDS^{-1}$. This allows us to compute successive powers of A easily:

$$A^2 = SDS^{-1}SDS^{-1} = SDIDS^{-1} = SD^2S^{-1}$$
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