Symmetric Matrices

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MASC

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Thus, everything in section 6.4 and following that refers to A^H and/or Hermitian matrices is valid for A^T and symmetric matrices, provided A has only real number entries.

Theorem

Let A be a symmetric $n \times n$ matrix, and suppose λ_1 and λ_2 are two different eigenvalues. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are the respective eigenvectors then $\mathbf{x}_1 \perp \mathbf{x}_2$.

To see this, consider

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\begin{aligned}
\lambda_1 \mathbf{x}_1^T \mathbf{x}_2 &= (\lambda_1 \mathbf{x}_1)^T \mathbf{x}_2 = (A\mathbf{x}_1)^T \mathbf{x}_2 \\
&= \mathbf{x}_1^T A^T \mathbf{x}_2 = \mathbf{x}_1^T A \mathbf{x}_2 \\
&= \mathbf{x}_1^T (\lambda_2) \mathbf{x}_2 = \lambda_2 \mathbf{x}_1^T \mathbf{x}_2
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\end{aligned}$$

By subtraction, $(\lambda_1 - \lambda_2)\mathbf{x}_1^T \mathbf{x}_2 = 0$. Since $\lambda_1 - \lambda_2 \neq 0$, we conclude that $\mathbf{x}_1^T \mathbf{x}_2 = 0$.

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Let A be a symmetric $n \times n$ matrix, and suppose $\mathbf{x}$ is an eigenvector for A. If $\mathbf{y} \perp \mathbf{x}$ then $A\mathbf{y} \perp \mathbf{x}$.

If λ is the eigenvalue associated to $\mathbf{x}$, then

$$(A\mathbf{y})^T\mathbf{x} = \mathbf{y}^T A^T\mathbf{x} = \mathbf{y}^T A\mathbf{x} = \lambda\mathbf{y}^T\mathbf{x} = 0.$$

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Whence, $\lambda^2 \geq 0$ and so λ must be real.

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To prove that A has a basis of eigenvectors, start with any eigenvector $\mathbf{x}_1$ with eigenvalue λ_1. We can arrange for $\|\mathbf{x}_1\| = 1$. Let $S = \mathrm{Span}(\mathbf{x}_1)^\perp$. Let $\mathbf{q}_2, \mathbf{q}_3, \ldots \mathbf{q}_n$ be an orthonormal basis for S. Consider the orthogonal matrix $Q = \begin{pmatrix} \mathbf{x}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{pmatrix}$. Now, because $A\mathbf{q}_j \perp \mathbf{x}_1$ for each j,

$$Q^T AQ = \left(\begin{array}{c|c} \lambda_1 & \mathbf{0} \\ \hline \mathbf{0} & B \end{array}\right),$$

where B is an $n-1 \times n-1$ symmetric matrix.

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We can repeat this process for $\mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2)^{\perp}$ to get another eigenvector, and so on. This works until there are no more nonzero vectors in $\mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots)^{\perp}$.

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Example: Find an orthonormal basis of eigenvectors for $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

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Eigenvalues: $\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)((2-\lambda)^2 - 1) = (2-\lambda)(\lambda^2 - 4\lambda + 3).$

Equate to zero to get $\lambda = 1, 2, 3$.

For $\lambda = 1$, the matrix

$$A - I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

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A basis for the eigenspace is $\begin{pmatrix} 0 & -1 & 1 \end{pmatrix}^T$. Normalize to get $\mathbf{q}_1 = \begin{pmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}^T$.

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$$A - 2I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

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A basis for the eigenspace is $\mathbf{q}_2 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$. The norm is already 1. Note that $\mathbf{q}_1 \perp \mathbf{q}_2$.

For $\lambda = 3$ the matrix

$$A - 3I = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{\text{3 EROs}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

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A basis for the eigenspace is $\begin{pmatrix} 0 & 1 & 1 \end{pmatrix}^T$. Normalize to get $\mathbf{q}_3 = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}^T$. Note that both $\mathbf{q}_1 \perp \mathbf{q}_3$ and $\mathbf{q}_2 \perp \mathbf{q}_3$.

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For example: let $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ For this matrix $\det(A - \lambda I) = -(\lambda - 1)^2(\lambda - 4)$, giving eigenvalues $\lambda = 1, 1, 4$.

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For $\lambda = 1$, the matrix $A - I$ is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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$$\begin{pmatrix} -1 & 1 & 0 \end{pmatrix}^T \text{ and } \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}^T$$

These are not orthogonal, though they are a basis for the eigenspace. If we want an orthonormal basis we can apply the Gram-Schmidt process to these two to get

$$\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{pmatrix}^T$$

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