Eigenvalues and Diagonalization

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MASC

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Some Examples

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Find $\det(A - \lambda I)$

$$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 1 & 3-\lambda & 1 \\ 2 & -1 & 1-\lambda \end{vmatrix} = (3-\lambda)(\lambda^2 - 4\lambda + 4)$$

Equate this to 0 and solve to get $\lambda = 3, 2, 2.$

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For $\lambda = 3$, the matrix

$$A - 3I = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & -1 & -2 \end{pmatrix} \quad \text{reduces to} \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

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(3) We can conclude that A is not diagonalizable.

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The determinant of $B - \lambda I$ is

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