Diagonalization

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MASC

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In fact, this case is the only way a matrix can be diagonalizable:

Theorem

An $n \times n$ matrix A is diagonalizable if and only if $\mathbb{R}^n$ has a basis consisting of eigenvectors of A. It happens when the sum of the dimensions of the eigenspaces is n. In that case the diagonal elements are the eigenvalues of A.

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Finally, for S to be invertible, its columns must be independent and so they are a basis for $\mathbb{R}^n$.

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We already have the information needed to do this:

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To check this, find S^{-1} and check that $D = S^{-1}AS$:

$$\begin{aligned} \begin{pmatrix} -3/5 & 2/5 \\ 1/5 & 1/5 \end{pmatrix} \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix} &= \begin{pmatrix} -3/5 & 2/5 \\ 1/5 & 1/5 \end{pmatrix} \begin{pmatrix} -0.5 & 2 \\ 0.5 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

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First step: find the eigenvalues.

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (2-\lambda)((1-\lambda)^2 - 1) = (2-\lambda)(\lambda^2 - 2\lambda)$$

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Equating this to 0 we get two roots: $\lambda = 0$ and a double root $\lambda = 2$.

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$$A - 2I = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \xrightarrow{\text{5 EROs}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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The equations this gives are $x_1 = x_3$ and $x_2 = 0$. The eigenspace is therefore

$$\left\{ \begin{pmatrix} \alpha \\ 0 \\ \alpha \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\} \quad \text{with basis} \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

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There is one special case where a matrix is assured to be diagonalizable: when there are as many different eigenvalues as the dimension.

This is because each eigenspace has dimension at least one, and in this case there will be n eigenspaces.