Eigenvalues and Eigenvectors

D. H. Luecking

MASC

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$$\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} [\mathbf{x}]_{\mathcal{B}} = [A\mathbf{x}]_{\mathcal{B}}.$$

Another way to put this is that if $S = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ then

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Precisely what one needs are vectors $\mathbf{x}$ that are not $\mathbf{0}$ and satisfy $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ.

Definition

If A is an $n \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$ is a nonzero vector such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ then $\mathbf{x}$ is called an *eigenvector* for A and λ is an *eigenvalue*. For a given eigenvalue λ, the set of solutions of $A\mathbf{x} = \lambda\mathbf{x}$ is called the *eigenspace associated to λ*.

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Every eigenvector has an associated eigenvalue. It turns out to be easier to find the eigenvalues first.

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If λ is an eigenvalue associated to an eigenvector $\mathbf{x}$, then $A - \lambda I$ must be singular and therefore $\det(A - \lambda I) = 0$.

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This theorem gives us the means to find eigenvalues: Equate $\det(A - \lambda I)$ to zero and solve for λ.

Let's apply this to the matrix $A = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}$

We get

$$\det(A - \lambda I) = \begin{vmatrix} 0.7 - \lambda & 0.2 \\ 0.3 & 0.8 - \lambda \end{vmatrix} = \lambda^2 - 1.5\lambda + 0.5 = (\lambda - 1)(\lambda - 0.5)$$

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Thus the eigenvalues are $\lambda = 1$ and $\lambda = 1/2$. To get an eigenvector for $\lambda = 1$, we need to solve $(A - I)\mathbf{x} = 0$:

$$(A - I)\mathbf{x} = \begin{pmatrix} -0.3 & 0.2 \\ 0.3 & -0.2 \end{pmatrix} \mathbf{x} = \mathbf{0}$$

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Row-reducing leads to $x_1 = (2/3)x_2$ and so the null space of $A - I$ is

$$\left\{ \begin{pmatrix} (2/3)\alpha \\ \alpha \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$$

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Any nonzero vector in this set is an eigenvector with eigenvalue $\lambda = 1$. Selecting $\alpha = 3$ gives us $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ but any other convenient multiple is also an eigenvector.

For eigenvalue $\lambda = 1/2$ we do the same to $A - (1/2)I$:

$$(A - (1/2)I)\mathbf{x} = \left(\begin{array}{cc} 0.2 & 0.2 \\ 0.3 & 0.3 \end{array} \right) \mathbf{x} = 0$$

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If $A\mathbf{x} = \lambda\mathbf{x}$ then $\alpha A\mathbf{x} = \alpha\lambda\mathbf{x}$, and $A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda(\lambda\mathbf{x})$.

Theorem

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ *is a basis for the eigenspace of* A *associated to* λ_1 *and* $\mathbf{v}_{r+1}, \ldots, \mathbf{v}_m$ *is a basis for the eigenspace of* A *associated to* λ_2 *with* $\lambda_2 \neq \lambda_1$*, then* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ *is independent.*

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Suppose

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Multiply this by λ_1 to get

$$\lambda_1(c_1\mathbf{v}_1 + \cdots + c_r\mathbf{v}_r) + \lambda_1(c_{r+1}\mathbf{v}_{r+1} + \cdots + c_m\mathbf{v}_m) = \mathbf{0}$$

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Subtract the last two equations to get

$$(\lambda_1 - \lambda_2)(c_{r+1}\mathbf{v}_{r+1} + \cdots + c_m\mathbf{v}_m) = \mathbf{0}$$

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By independence, $c_{r+1} = c_{r+2} = \cdots = c_m = 0$. That means the first equation becomes

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Ideally, we want a basis of eigenvectors. If we have that, say $\mathcal{B} = [\mathbf{v}_1, \mathbf{v}_1, \ldots, \mathbf{v}_n]$ is a basis and $A\mathbf{v}_j = \lambda_j\mathbf{v}_j$ for every j, then we have the following:

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$[A\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 c_1 \\ \vdots \\ \lambda_n c_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} [\mathbf{x}]_{\mathcal{B}}$. Moreover, if

$S = \begin{pmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_n \end{pmatrix}$ (the transition matrix from $\mathcal{B}$ to $\mathcal{E}$), then $S^{-1}AS$ is a diagonal matrix, with eigenvalues of A on the diagonal.

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Find the eigenspaces for these eigenvalues.

The nullspace of $A - 2I$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{\text{2 EROs}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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So we get a basis for this eigenspace: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

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Then x_1 and x_2 are the leading variables and x_3 is free. The equations are $x_1 = 0$ and $x_2 = -x_3$. This gives the eigenspace

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Because we have altogether 3 independent vectors, we have a basis for $\mathbb{R}^3$ consisting of eigenvectors. If we put the basis vectors in a matrix S and find its inverse S^{-1}, then the product $S^{-1}AS$ will be a diagonal matrix D with $2, 2, 1$ on the diagonal:

That is, $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$, $S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

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Note that $A = SDS^{-1}$. This allows us to compute successive powers of A easily:

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