More Orthogonal Stuff

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MASC

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Moreover, $Q = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$ is an orthogonal matrix and $Q^T = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$ is its inverse.

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Therefore, $\mathcal{B} = \left[\mathbf{u}_1 = (1/\sqrt{6})\mathbf{v}_1, \mathbf{u}_2 = (1/\sqrt{3})\mathbf{v}_2, \mathbf{u}_3 = (1/\sqrt{2})\mathbf{v}_3\right]$ is an orthonormal basis and

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$$Q = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{3} & 0 \end{pmatrix}$$

is an orthogonal matrix.

Taking any vector, for example $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ we get

$$[\mathbf{v}]_{\mathcal{B}} = Q^T \mathbf{v} = \begin{pmatrix} \mathbf{u}_1^T \mathbf{v} \\ \mathbf{u}_2^T \mathbf{v} \\ \mathbf{u}_3^T \mathbf{v} \end{pmatrix} = \begin{pmatrix} 5/\sqrt{6} \\ 2/\sqrt{3} \\ 1/\sqrt{2} \end{pmatrix}$$

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Recall: If U is the matrix whose columns are $\mathbf{u}_1, \ldots, \mathbf{u}_r$, then we solve $U^T U \mathbf{x} = U^T \mathbf{b}$ to get $\hat{\mathbf{x}}$,

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Recall: If U is the matrix whose columns are $\mathbf{u}_1, \ldots, \mathbf{u}_r$, then we solve $U^TU\mathbf{x} = U^T\mathbf{b}$ to get $\hat{\mathbf{x}}$, and then $U\hat{\mathbf{x}}$ is then the closest vector to $\mathbf{b}$ in the column space of U (which is S).

But, since U has orthogonal columns, the $U^T U = I$ $(r \times r)$ and so the solution of $U^T U\mathbf{x} = U^T\mathbf{b}$ is $\hat{\mathbf{x}} = U^T\mathbf{b}$ and the closest vector in S to $\mathbf{b}$ is $U\hat{\mathbf{x}} = UU^T\mathbf{b}$.

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Here is an example. Find the projection of $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ onto the span of $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

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$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

These are orthogonal, but not orthonormal, so we "normalize" them:

$\mathbf{u}_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, and we get the projection matrix

$$P = UU^T = \begin{pmatrix} 1/3 & 0 \\ 2/3 & -1/\sqrt{2} \\ 2/3 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1/9 & 2/9 & 2/9 \\ 2/9 & 17/18 & -1/18 \\ 2/9 & -1/18 & 17/18 \end{pmatrix}$$

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We can check our work (in part) by determining whether the difference $\mathbf{v} - P\mathbf{v} = \begin{pmatrix} 2/3 \\ -1/6 \\ -1/6 \end{pmatrix}$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$.