Orthogonal Bases

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Definition

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Simply set $c_1 = 1$ and the rest of the $c_j = 0$.

Now suppose $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is orthogonal and that

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Theorem

Every nonzero subspace of $\mathbb{R}^n$ has a basis that is orthonormal.

To see this, let S be a subspace. It is enough to find an orthogonal spanning set in S: it will automatically be independent. Then we can replace each vector $\mathbf{v}_j$ with $(1/\left\|\mathbf{v}_j\right\|)\mathbf{v}_j$ to get an orthonormal basis.

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We can repeat this until we have an orthogonal set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ in S with $r = \dim S$. Since the set is independent and its size is $\dim S$, it must also be spanning and so is a basis.

Theorem

Let S is a subspace of $\mathbb{R}^n$ and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be an orthogonal basis for S. If $\mathbf{v}$ is any vector in $\mathbb{R}^n$ and $\mathbf{p}$ the element of S that is closest to $\mathbf{v}$, then

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Notice that each term in the sum $\sum_{j=1}^{r} \dfrac{\mathbf{v}^T \mathbf{v}_j}{\mathbf{v}_j^T \mathbf{v}_j} \mathbf{v}_j$ has the same formula as the vector projection of $\mathbf{v}$ onto $\mathbf{v}_j$.

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To see why this is true, recall that the closest vector $\mathbf{p}$ is the one that makes $\mathbf{v} - \mathbf{p} \perp S$.

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To see why this is true, recall that the closest vector $\mathbf{p}$ is the one that makes $\mathbf{v} - \mathbf{p} \perp S$. So we only need to check that this formula makes $\mathbf{v} - \mathbf{p} \perp \mathbf{v}_i$ for each of the basis vectors $\mathbf{v}_i$.

$$\begin{aligned}
\mathbf{v}_i^T(\mathbf{v}-\mathbf{p}) &= \mathbf{v}_i^T\mathbf{v} - \mathbf{v}_i^T \sum_{j=1}^{r} \frac{\mathbf{v}^T\mathbf{v}_j}{\mathbf{v}_j^T\mathbf{v}_j}\,\mathbf{v}_j \\
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\end{aligned}$$

Some key properties of the scalar product:

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We have seen that, because of the first of these, we can use $\|\mathbf{v}\| = (\mathbf{v}^T\mathbf{v})^{1/2}$ as a measure of the size of $\mathbf{v}$. Moreover, in the special cases $\mathbb{R}^2$ and $\mathbb{R}^3$, $\|\mathbf{x}-\mathbf{y}\|$ is the distance between the points with coordinates $\mathbf{x}$ and $\mathbf{y}$.

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This suggests the following definition.

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If V is a vector space, then an *inner product* is an operation that assigns, to any pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in V, a real number $\langle \mathbf{x}, \mathbf{y} \rangle$ satisfying

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Theorem (Pythagorean Formula)

If $\mathbf{x} \perp \mathbf{y}$ *then* $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

Definition

If V is a vector space with an inner product $\langle \mathbf{x}, \mathbf{y} \rangle$, then $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$. This is called the *norm induced by* this inner product.

Definition

If V is a vector space with an inner product $\langle \mathbf{x}, \mathbf{y} \rangle$, then we say $\mathbf{x}$ *is orthogonal to* $\mathbf{y}$ if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. and we express this by $\mathbf{x} \perp \mathbf{y}$.

Theorem (Pythagorean Formula)

If $\mathbf{x} \perp \mathbf{y}$ *then* $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

We prove this by

$$\begin{aligned} \|\mathbf{x} + \mathbf{y}\|^2 &= \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} + \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle \\ &= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \end{aligned}$$

Theorem (Cauchy-Schwarz Inequality)

For any $\mathbf{x}$ *and* $\mathbf{y}$ *in an inner product space,*

$$| \langle \mathbf{x}, \mathbf{y} \rangle | \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$$

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Otherwise consider the following for any real numbers α and β

$$\begin{aligned}
0 &\leq \langle \alpha\mathbf{x} - \beta\mathbf{y}, \alpha\mathbf{x} - \beta\mathbf{y} \rangle \\
0 &\leq \langle \alpha\mathbf{x}, \alpha\mathbf{x} \rangle + \langle \alpha\mathbf{x}, -\beta\mathbf{y} \rangle + \langle -\beta\mathbf{y}, \alpha\mathbf{x} \rangle + \langle -\beta\mathbf{y}, -\beta\mathbf{y} \rangle \\
0 &\leq \alpha^2 \langle \mathbf{x}, \mathbf{x} \rangle - \alpha\beta \langle \mathbf{x}, \mathbf{y} \rangle - \alpha\beta \langle \mathbf{y}, \mathbf{x} \rangle + \beta^2 \langle \mathbf{y}, \mathbf{y} \rangle \\
0 &\leq \alpha^2 \|\mathbf{x}\|^2 - 2\alpha\beta \langle \mathbf{x}, \mathbf{y} \rangle + \beta^2 \|\mathbf{y}\|^2
\end{aligned}$$

Altogether we get

$$\alpha\beta \langle \mathbf{x}, \mathbf{y} \rangle \leq (1/2)(\alpha^2 \|\mathbf{x}\|^2 + \beta^2 \|\mathbf{y}\|^2)$$

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Since this holds for any α, β, let $\alpha = 1/\|\mathbf{x}\|$ and $\beta = 1/\|\mathbf{y}\|$ to get

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Theorem (Triangle Inequality)

For any pair of vectors $\mathbf{x}$ *and* $\mathbf{y}$, $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

The proof relies on the Cauchy-Schwarz Inequality:

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$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|^2 &= \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle \\ &= \|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y}\rangle + \|\mathbf{y}\|^2 \\ &\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\,\|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2\end{aligned}$$

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Then take square roots to get $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$.

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&\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\,\|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2
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Then take square roots to get $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

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Then take square roots to get $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

This allows us to express the distance between vectors by $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|$ and deduce that

$$\begin{aligned}d(\mathbf{x}, \mathbf{z}) &= \|\mathbf{x}-\mathbf{z}\| = \|\mathbf{x}-\mathbf{y}+\mathbf{y}-\mathbf{z}\| \\ &\leq \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}-\mathbf{z}\| = d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})\end{aligned}$$