Orthogonal Subspaces

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MASC

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The first condition is the statement that $\mathbf{b} \in \mathcal{R}(A)$. The second condition is that $\mathbf{b}$ is not in $\mathcal{N}(A^T)^\perp$.

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Because $\mathbf{d} \in \mathcal{N}(A^T)$, if we apply A^T to $\mathbf{b} = \mathbf{c} + \mathbf{d}$ we get

$$A^T(\mathbf{b}) = A^T(\mathbf{c} + \mathbf{d}) = A^T\mathbf{c}$$

So, if we multiply the equation $A\mathbf{x} = \mathbf{b}$ (which has no solution) by A^T we get $A^T A\mathbf{x} = A^T\mathbf{b} = A^T\mathbf{c}$.

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Recall that we have written $\mathbf{b} = \mathbf{c} + \mathbf{d}$ where $\mathbf{c} \in \mathcal{R}(A)$ and $\mathbf{d} \perp \mathcal{R}(A)$. We will now see that the closest vector to $\mathbf{b}$ in $\mathcal{R}(A)$ is $\mathbf{c}$ and the distance from $\mathbf{b}$ to $\mathbf{c}$ is $\|\mathbf{d}\|$.

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(\mathbf{b} - \mathbf{v})^T(\mathbf{b} - \mathbf{v}) &= (\mathbf{d} + \mathbf{c} - \mathbf{v})^T(\mathbf{d} + \mathbf{c} - \mathbf{v}) \\
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Example: Suppose we perform an experiment where we measure the output y for various inputs x.

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Example: Suppose we perform an experiment where we measure the output y for various inputs x. That is we get a table of data of the form $\{(x_i, y_i) \mid i = 1, 2, \ldots, n\}$. Then we plot these points and try to find the best curve of some sort that matches this data.

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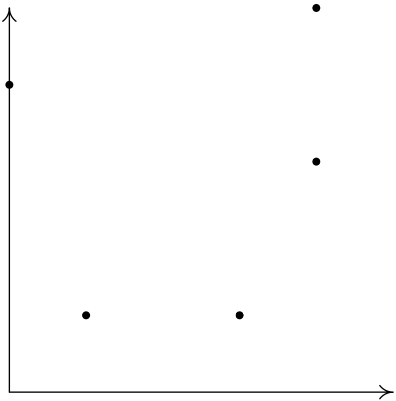

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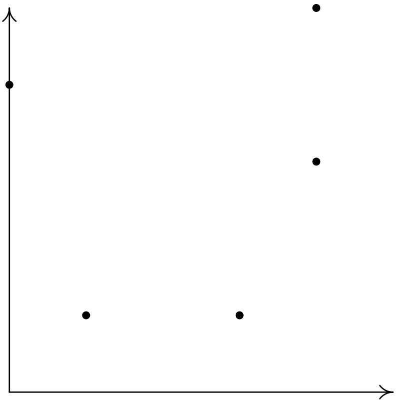

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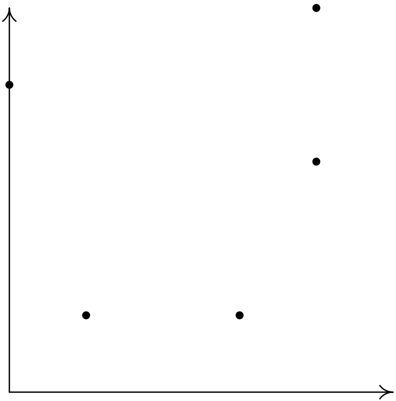

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The data can be used to determine what a, b, c must be. Ideally, we want the equations to $ax_i^2 + bx_i + c = y_i$ to hold for each data point.

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The equations for a, b, c are

$$\begin{aligned} c &= 4 \\ 1a + 1b + c &= 2 \\ 9a + 3b + c &= 1 \\ 16a + 4b + c &= 3 \\ 16a + 4b + c &= 5 \end{aligned}$$

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or $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \\ 16 & 4 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 4 \\ 2 \\ 1 \\ 3 \\ 5 \end{pmatrix},$$

By the previous argument, we get the vector $\mathbf{x}$ that makes $A\mathbf{x}$ closest to $\mathbf{b}$ by solving $A^T A\mathbf{x} = A^T\mathbf{b}$.

So we compute

$$A^T A = \begin{pmatrix} 594 & 156 & 42 \\ 156 & 42 & 12 \\ 42 & 12 & 5 \end{pmatrix} \quad \text{and} \quad A^T\mathbf{b} = \begin{pmatrix} 139 \\ 37 \\ 15 \end{pmatrix}$$

and row-reduce

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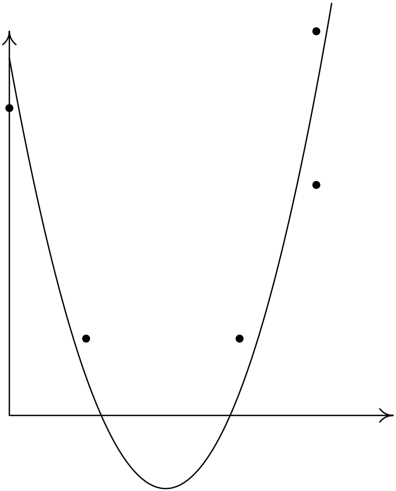
to get (approximately) $a = 1.35$, $b = -5.5$ and $c = 4.65$. I have graphed the quadratic $1.35x^2 - 5.5x + 4.65$ on the same plot as the data points on the next slide.



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Multiplying by the transpose gives

$$\begin{pmatrix} 9 & -7 \\ -7 & 11 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 1 \\ -2 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

Multiplying by the transpose gives

$$\begin{pmatrix} 9 & -7 \\ -7 & 11 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

This has a solution $\hat{\mathbf{x}} = \begin{pmatrix} 83/50 \\ 71/50 \end{pmatrix} = \begin{pmatrix} 1.66 \\ 1.42 \end{pmatrix}$.

If we want to see how close we have come, we find

$$A\hat{\mathbf{x}} = \begin{pmatrix} 3.08 \\ 0.94 \\ 1.9 \end{pmatrix}$$

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In any problem $A\mathbf{x} = \mathbf{b}$, the difference $r(\mathbf{x}) = \mathbf{b} - A\mathbf{x}$ is called the *residual vector* assosiated to $\mathbf{x}$. The least squares solution, is a vector $\hat{\mathbf{x}}$ that gives the residual vector the smallest possible norm.

Using the method of least squares to get vectors orthogonal to a subspace.

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If S is a subspace of $\mathbb{R}^n$ and $\mathbf{b}$ is a vector in $\mathbb{R}^n$, we know that there exist unique vectors $\mathbf{u} \in S$ and $\mathbf{v} \in S^\perp$ such that $\mathbf{b} = \mathbf{u} + \mathbf{v}$.

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If we apply this to A^T in place of A we can split $\mathbf{b}$ into $\mathbf{u}$ in $\mathcal{R}(A^T) = \mathcal{N}(A)^\perp$ and $\mathbf{v}$ in $\mathcal{R}(A^T)^\perp = \mathcal{N}(A)$.

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That is, we solve $AA^T\mathbf{x} = A\mathbf{b}$ to get $\hat{x}$ and then $\mathbf{u} = A^T\hat{x}$ is in $\mathcal{N}(A)^\perp$ and $\mathbf{v} = \mathbf{b} - A^T\hat{x}$ is in $\mathcal{N}(A)$.

Example: Let S be the span of $(1,1,2,0)^T$ and $(0,1,2,-2)^T$ and let $\mathbf{b}=(1,1,1,1)^T$. Find the vectors $\mathbf{u}\in S$ and $\mathbf{v}\in S^\perp$ such that $\mathbf{b}=\mathbf{u}+\mathbf{v}$.

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Now S is the column space of the 4×2 matrix A below and we need the least squares solution of

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 2 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

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Multiplying by A^T we get

$$\begin{pmatrix} 6 & 5 \\ 5 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

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$\mathbf{u} = A\hat{\mathbf{x}} = (31/29, 17/29, 34/29, 28/29)^T$ and
$\mathbf{v} = \mathbf{b} - A\hat{\mathbf{x}} = (-2/29, 12/29, -5/29, 1/29)$.