Orthogonal Subspaces

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Theorem

If S is a subspace of $\mathbb{R}^n$ then $\dim S + \dim S^\perp = n$. Furthermore, if $\{\mathbf{x}_1, \ldots, \mathbf{x}_r\}$ is a basis for S and $\{\mathbf{x}_{r+1}, \ldots, \mathbf{x}_n\}$ is a basis for $S^\perp$, then $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a basis for $\mathbb{R}^n$.

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To see that the two bases for S and $S^\perp$ together form a basis for $\mathbb{R}^n$, we first show it is independent.

To check independence, consider a linear combination

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We want to show that all the c_i are zero. Set

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_r\mathbf{x}_r \quad \text{and} \quad \mathbf{y} = c_{r+1}\mathbf{x}_{r+1} + c_{r+2}\mathbf{x}_{r+2} + \cdots + c_n\mathbf{x}_n.$$

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Since the set $\{\mathbf{x}_1, \ldots \mathbf{x}_n\}$ has n elements and is independent, it must also span $\mathbb{R}^n$, so it is a basis of $\mathbb{R}^n$

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To see that $\mathbf{u}$ and $\mathbf{v}$ are unique, suppose there were another pair $\mathbf{u}' \in S$ and $\mathbf{v}' \in S^\perp$ and $\mathbf{x} = \mathbf{u}' + \mathbf{v}'$. Then

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But $\mathbf{v}^T\mathbf{v} = 0$ implies $\mathbf{v} = \mathbf{0}$. Therefore, $\mathbf{x} = \mathbf{u} + \mathbf{0} \in S$.

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This last one provides another answer to the question: for which vectors $\mathbf{b}$ does the equation $A\mathbf{x} = \mathbf{b}$ have a solution? The existence of $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$ means that $\mathbf{b}$ is in the column space of A. By the above, $\mathbf{b}$ must be orthogonal to the null space of A^T. If we find a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ for the null space of A^T, then the condition for $A\mathbf{x} = \mathbf{b}$ to have a solution is that $\mathbf{v}_1^T\mathbf{b} = 0, \dots, \mathbf{v}_k^T\mathbf{b} = 0$.

Example: For what vectors $\mathbf{b}$ does the following have a solution?

$$\begin{aligned} 2x_1 + 2x_2 + 4x_3 &= b_1 \\ x_1 + \qquad + \ x_3 &= b_2 \\ x_2 + \ x_3 &= b_3 \end{aligned}$$

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$$\begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \\ 4 & 1 & 1 \end{pmatrix} \xrightarrow{\text{6 EROs}} \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{i.e.} \quad \begin{cases} x_1 + (1/2)x_3 = 0 \\ x_2 - x_3 = 0 \end{cases}$$

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Then $\mathcal{N}(A^T)$ is spanned by one vector $\left(\begin{array}{c} -1/2 \\ 1 \\ 1 \end{array}\right)$. So, we conclude that a solution exists if and only if

$$\left(\begin{array}{ccc} -1/2 & 1 & 1 \end{array}\right) \mathbf{b} = -(1/2)b_1 + b_2 + b_3 = 0$$