Scalar Product
(cont.)

D. H. Luecking

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Then the distance from a point $P = (x, y)$ to the line would be the length of the projection of $\overrightarrow{P_0P}$ onto the perpendicular vector. That is,

$$\text{distance } = \left| \frac{a(x - x_0) + b(y - y_0)}{(a^2 + b^2)^{1/2}} \right|$$

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If $P = (x, y, z)$ is a point and

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is the equation of a plane that contains the point $P_0 = (x_0, y_0, z_0)$, then the distance from P to that plane is the length of the projection of $\mathbf{x} = \overrightarrow{P_0P}$ onto the perpendicular vector $\mathbf{a} = \left(\begin{array}{c} a \\ b \\ c \end{array}\right)$. This is the absolute value of the scalar projection.

$$\left|\frac{\mathbf{x}^T\mathbf{a}}{\|\mathbf{a}\|}\right| = \left|\frac{a(x-x_0)+b(y-y_0)+c(z-z_0)}{(a^2+b^2+c^2)^{1/2}}\right|$$

To get the point Q on the plane that is nearest to P we use

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lie in the plane. We can find a vector perpendicular to the plane by finding a vector orthogonal to both $\mathbf{a}_1$ and $\mathbf{a}_2$. We saw how to do this earlier (with exactly these two vectors). One such perpendicular vector is $\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$

Then one equation for the plane is obtained using P_0 and this perpendicular vector:

$$-5(x-1) - 2(y-0) + (z-1) = 0 \quad \text{or} \quad -5x - 2y + z + 4 = 0$$

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For example, in $\mathbb{R}^3$, both the vector $\mathbf{e}_1, \mathbf{e}_2$ are orthogonal to $\mathbf{e}_3$. Thus, the following are orthogonal subspaces

$$V = \left\{ \left(\begin{array}{c} \alpha \\ \beta \\ 0 \end{array} \right) \middle| \ \alpha, \beta \in \mathbb{R} \right\} \text{ and } W = \left\{ \left(\begin{array}{c} 0 \\ 0 \\ \gamma \end{array} \right) \middle| \ \gamma \in \mathbb{R} \right\}$$

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If S is a subspace of $\mathbb{R}^n$ then the *orthogonal complement of S* is the set of all vectors that are orthogonal to every vector in S. We denote this set $S^\perp$. Formally:

$$S^\perp = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}^T\mathbf{y} = 0 \text{ for all } \mathbf{y} \in S\}.$$

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1. If V and W are orthogonal subspaces of $\mathbb{R}^n$ then the only vector that belongs to both V and W is $\mathbf{0}$. Proof: if $\mathbf{x} \in V$ and $\mathbf{x} \in W$ then by orthogonality $\mathbf{x}^T\mathbf{x} = 0$. But we have seen this means $\mathbf{x} = \mathbf{0}$.

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2. For any subspace S, $S^\perp$ is also a subspace. Proof of closure under addition: if $\mathbf{x}_1 \in S^\perp$ and $\mathbf{x}_2 \in S^\perp$, and if $\mathbf{y}$ is any vector in S then $\mathbf{y}^T(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{y}^T\mathbf{x}_1 + \mathbf{y}^T\mathbf{x}_2 = 0 + 0 = 0$. This means $\mathbf{x}_1 + \mathbf{x}_2 \in S^\perp$.

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We will see how to calculate the orthogonal complement of both of these kinds of subspaces.

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