Scalar Product

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15 March 2024

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In $\mathbb{R}^2$ we have $\|\mathbf{x}\| = (x_1^2 + x_2^2)^{1/2}$, and in $\mathbb{R}^3$ we have $\|\mathbf{x}\| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. In either case this is the length of the arrow that we use to visualize $\mathbf{x}$.

A couple of simple properties of scalar products.

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Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ or $\mathbb{R}^3$. Suppose $\|\mathbf{x}\| \neq 0$ and $\|\mathbf{y}\| \neq 0$. Let θ be the angle between $\mathbf{x}$ and $\mathbf{y}$ with $0 \leq \theta \leq 180^\circ$. Then

$$\cos\theta = \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}.$$

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We're actually going to prove this in the case $\mathbb{R}^2$. But first recall the linear transformations on $\mathbb{R}^2$ given by rotations. That is, if γ is any angle let $R_\gamma\mathbf{x}$ be the arrow $\mathbf{x}$ rotated counterclockwise by the angle γ.

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Similarly

$$\|R_\gamma \mathbf{x}\| = \big((R_\gamma \mathbf{x})^T (R_\gamma \mathbf{x})\big)^{1/2} = (\mathbf{x}^T \mathbf{x})^{1/2} = \|\mathbf{x}\|$$

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Example: if $\mathbf{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ then $\|\mathbf{x}\| = 5$ and $\|\mathbf{y}\| = \sqrt{2}$, $\mathbf{x}^T\mathbf{y} = 1$.

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$$\cos\theta = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}^T \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{-3}{5\sqrt{2}} + \frac{4}{5\sqrt{2}} = \frac{1}{5\sqrt{2}}$$

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Theorem

If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$, then $|\mathbf{x}^T\mathbf{y}| \le \|\mathbf{x}\|\,\|\mathbf{y}\|$.

In $\mathbb{R}^2$ or $\mathbb{R}^3$ this follows from $\mathbf{x}^T\mathbf{y} = \|\mathbf{x}\|\,\|\mathbf{y}\| \cos\theta$ and the fact that $|\cos\theta| \le 1$. In $\mathbb{R}^n$ for $n > 3$ we can rotate 2 variables at a time (without changing the scalar product or the norms) until coordinates x_j and y_j are 0 for all $j > 2$. Then apply the $\mathbb{R}^2$ case.

Definition

If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{R}^n$, we say that $\mathbf{x}$ *is orthogonal to* $\mathbf{y}$ if $\mathbf{x}^T\mathbf{y} = 0$. We denote this by writing $\mathbf{x} \perp \mathbf{y}$.

Note that if $\mathbf{x} \perp \mathbf{y}$ then also $\mathbf{y} \perp \mathbf{x}$. Also $\mathbf{0}$ is orthogonal to any vector. In $\mathbb{R}^2$ or $\mathbb{R}^3$, if neither $\mathbf{x}$ nor $\mathbf{y}$ is $\mathbf{0}$ then $\mathbf{x} \perp \mathbf{y}$ means that the angle between them is $90°$.

Some properties of orthogonality:

1. If $\mathbf{x} \perp \mathbf{y}_1, \mathbf{x} \perp \mathbf{y}_2, \ldots, \mathbf{x} \perp \mathbf{y}_n$ then $\mathbf{x}$ is orthogonal to every vector in $\mathrm{Span}(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n)$. Proof:

$$\mathbf{x}^T(c_1\mathbf{y}_1 + \cdots + c_n\mathbf{y}_n) = c_1\mathbf{x}^T\mathbf{y}_1 + \cdots + c_n\mathbf{x}^T\mathbf{y}_n = 0 + \cdots + 0 = 0$$

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Here x_1 is leading, x_2 and x_3 are free. The basic solutions

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

are orthogonal to $\mathbf{a}$, as is every vector in their span.

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$$\begin{pmatrix} 1 & -1 & 3 \\ -1 & 2 & -1 \end{pmatrix} \xrightarrow{R_2+R_1} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow{R_1+R_2} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \end{pmatrix}$$

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This has leading variables x_1 and x_2 with free variable x_3, also $x_1 = -5x_3$ and $x_2 = -2x_3$ and so the vectors $\begin{pmatrix} -5\alpha \\ -2\alpha \\ \alpha \end{pmatrix}$ are orthogonal to $\mathbf{a}_1$ and $\mathbf{a}_2$, for any choice of α.

Orthogonal Projection

If $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ then $\|\mathbf{x}\| = (x_1^2 + x_2^2)^{1/2}$ is the distance from the point $(0, 0)$ to (x_1, x_2).

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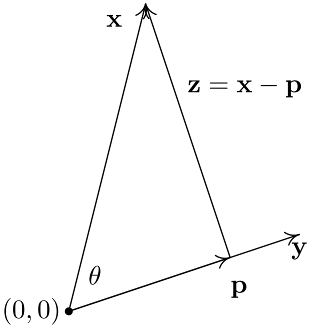
Suppose we want to find the point on a line that is closest to the point (x_1, x_2). Suppose we can express a line as all points that are tips of the vectors $\alpha\mathbf{y}$ for all $\alpha \in \mathbb{R}$.

The following is a picture of this setup:



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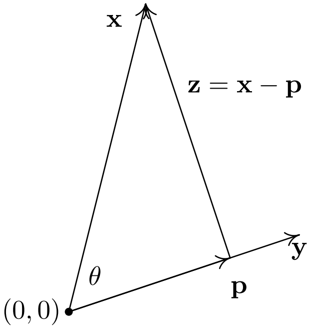

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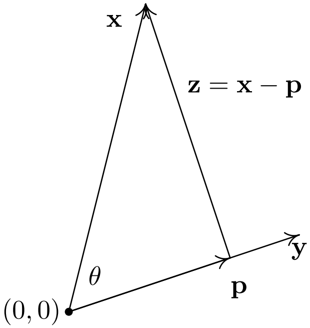

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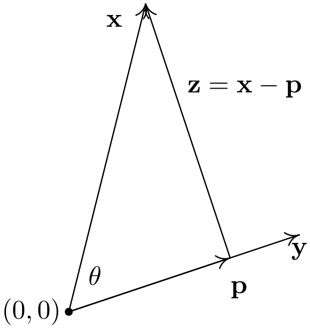

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To get $\mathbf{p}$ itself we observe that $\mathbf{u} = (1/\|\mathbf{y}\|)\mathbf{y}$ has the same direction as both $\mathbf{y}$ and $\mathbf{p}$, but has length 1.

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Definition

If $\mathbf{x}$ and $\mathbf{y}$ belong to $\mathbb{R}^2$ or $\mathbb{R}^3$ then:

The number $\alpha = \dfrac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{y}\|}$ is called the *scalar projection of $\mathbf{x}$ onto $\mathbf{y}$*.

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Note that the scalar projection α times the vector $\mathbf{u} = (1/\|\mathbf{y}\|)\mathbf{y}$ is the vector projection.

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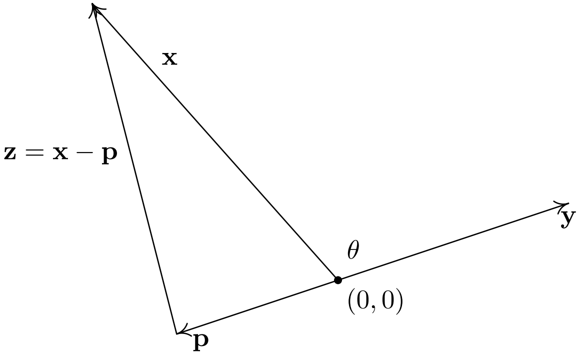

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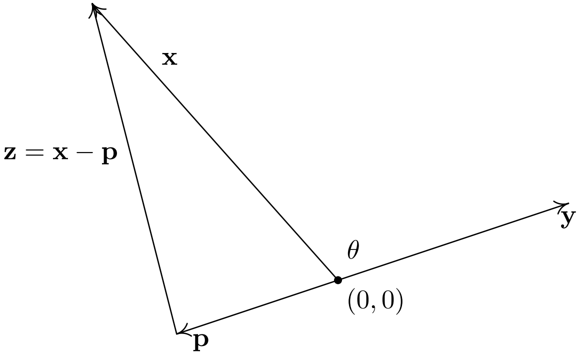

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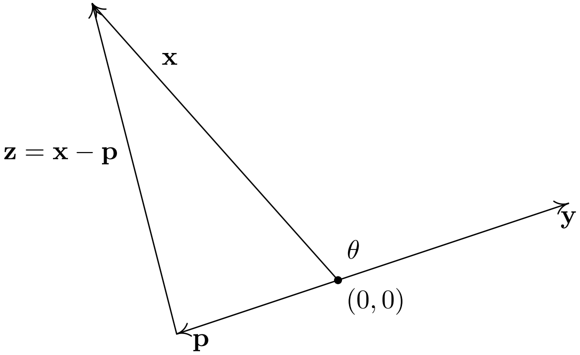

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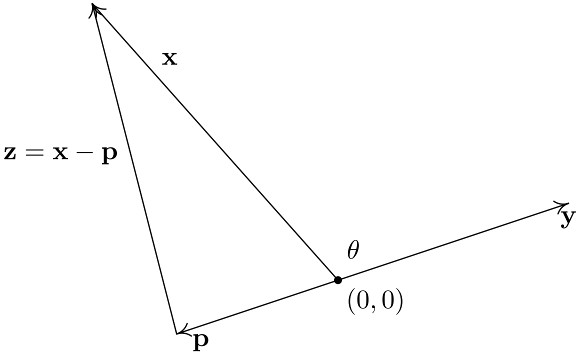

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We can take $\mathbf{y} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Consider the point $(1, 4)$, whose vector form is

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$$\mathbf{p} = \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y} = \frac{7}{10}\mathbf{y} = \begin{pmatrix} 2.1 \\ 0.7 \end{pmatrix}$$

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If we try this with a line L that does not pass through $(0, 0)$ we simply have to pick two points P_0 and P_1 on the line. The displacement vector from P_0 to P_1 will be denoted $\overrightarrow{P_0P_1}$. All the calculations to get the point on L closest to a point P then take place with $\mathbf{x} = \overrightarrow{P_0P}$ and $\mathbf{y} = \overrightarrow{P_0P_1}$. When we find $\mathbf{p}$, the closest point will be the point Q such that $\overrightarrow{P_0Q} = \mathbf{p}$.

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Say we want the closest point from $P = (1, 4)$ to this line. Then $\mathbf{x} = \overrightarrow{P_0P} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

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Say we want the closest point from $P = (1,4)$ to this line. Then $\mathbf{x} = \overrightarrow{P_0P} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Then we get the projection of $\mathbf{x}$ onto $\mathbf{y}$

$$\mathbf{p} = \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y} = \frac{6}{10}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 0.6 \end{pmatrix}$$

To get that closest point, we add the components of $\mathbf{p}$ to P_0 to get $Q = (1.8, 1.6)$ and the distance $((1-1.8)^2 + (4-1.6)^2)^{1/2} = \sqrt{6.4} \approx 2.53$.

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$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$