Review

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11 March 2024

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1. It is not closed under scalar multiplication.
2. It is not closed under addition.

The subset

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \;\middle|\; x_1 = x_2, \text{ and } x_2 = 2x_3 \right\}.$$

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$$\left\{ \left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right) \middle| \, x_1 = x_2, \text{ and } x_2 = 2x_3 \right\}.$$

is a subspace of $\mathbb{R}^3$ because it is the null space of the matrix

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is not a subspace of $\mathbb{R}^3$ because $\left(\begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right)$ is in this subset but the scalar multiple $\left(\begin{array}{c} 2 \\ 0 \\ 2 \end{array} \right)$ is not.

The set of all solutions of a homogeneous $n \times k$ system is a vector space, a subspace of $\mathbb{R}^k$. This is equivalent to being the null space of the system matrix.

Definition

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are vectors in a vector space V, a sum of the form

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars, is called a *linear combination of* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. The set of all such linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is called the *span of* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

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In $\mathbb{R}^n$, for $1 \leq j \leq n$, we let $\mathbf{e}_j$ be the vector that has a 1 in position j and zeros in every other position.

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Example 1: Do the following vectors span $\mathbb{R}^3$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix}$$

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Solution: a linear combination of these equated to any vector produces the following system matrix:

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 2 & -3 & -1 & 5 \\ 3 & 2 & 5 & 1 \end{pmatrix} \xrightarrow{\text{4 EROs}} \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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They do not span $\mathbb{R}^3$.

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A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is said to be *linearly dependent* if there exists scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, not all of which are zero, such that

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Determining whether a set of vectors is independent or dependent again amounts to a system of equations. But this time, it is not whether the system has a solution, but whether it has a nontrivial solution.

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Determining whether a set of vectors is independent or dependent again amounts to a system of equations. But this time, it is not whether the system has a solution, but whether it has a nontrivial solution.

If we want to know whether a set of vectors is independent, we convert the question to a system of equations and row-reduce the *system* matrix. If the echelon form indicates any free variables, the vectors are not independent.

Example: Are these vectors independent:

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \mathbf{w}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

The matrix is

$$\left(\begin{array}{ccc} 1 & 1 & 1 \\ 3 & 0 & 2 \\ 5 & 2 & 3 \end{array}\right) \xrightarrow{\text{Several EROs}} \left(\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{array}\right)$$

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Other shortcuts: (1) If the matrix is square, the set is either both independent and spanning, or neither. (2) If there are more columns than rows, the vectors are not independent (they may or may not be spanning). (3) If there are more rows than columns, the vectors are not spanning (they may or may not be independent).

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$$\mathbf{w} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$$

If the set is also independent, then there is only one choice of coefficients that will produce $\mathbf{w}$.

Standard bases

Some vector spaces have a basis so closely associated with the structure of the vectors that they are called *standard* bases.

The standard basis for $\mathbb{R}^n$ is the set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. For example, for $\mathbb{R}^3$ this basis is

$$\left(\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right), \quad \left(\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right), \quad \left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right)$$

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If a set of vectors $\{w_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ is independent, then any set of vectors with fewer than m elements cannot be spanning.

Theorem

Any two bases for a vector space have the same number of elements.

Definition

The dimension of a vector space is the number of elements in any basis.

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The dimension of $\mathbb{R}^{n \times k}$ is nk: $\{E_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$ is a basis.

Theorem

If the dimension of V is n then any set in V with more than n elements is dependent, and any set with fewer than n elements is not spanning.

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If the dimension of V is n then any set in V with more than n elements is dependent, and any set with fewer than n elements is not spanning. Any set with n elements either is both independent and spanning or is neither independent nor spanning.

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Example: Trim the following to an independent set with the same span

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

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Put these columns in a matrix and reduce to echelon form:

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & -1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{5 EROs}} \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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This tells us that the original set of vectors in spanning (no row of zeros) but is not independent.

But it also tells us that if we take a linear combination and equate it to $\mathbf{0}$:

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \alpha_4\mathbf{v}_4 = \mathbf{0}$$

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In general, if you keep the vectors corresponding to columns with leading 1s, (and discard the rest) you get an independent set with the same span.

Coordinates

Definition

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$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

the *coordinate vector relative to* $\mathcal{B}$. We denote it by $[\mathbf{v}]_{\mathcal{B}}$.

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There are several problems associated with bases and coordinates:

Given a vector $\mathbf{v}$ and an ordered basis $\mathcal{B}$, what is $[\mathbf{v}]_{\mathcal{B}}$?

Given a second basis $\mathcal{C}$, what is the relationship between $[\mathbf{v}]_{\mathcal{C}}$ and $[\mathbf{v}]_{\mathcal{B}}$?

The simplest cases are when $\mathcal{B}$ is one of the standard bases.

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If $\mathcal{E} = [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n]$ in $\mathbb{R}^n$ (interpreted as the space of column vectors) then $[\mathbf{v}]_{\mathcal{E}} = \mathbf{v}$.

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Suppose, in $\mathbb{R}^n$ we have another ordered basis $\mathcal{B} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$. Then finding $[\mathbf{v}]_{\mathcal{B}}$ amounts to solving

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{v} \implies [\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

The left side of this is the same as

$$\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{v}$$

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By the properties of a basis, this system has a solution for every choice of $\mathbf{v}$, which means this matrix

$$S = \left(\begin{array}{cccc} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{array} \right) \quad \text{is invertible.}$$

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We call S^{-1} the transition matrix from $\mathcal{E}$ to $\mathcal{B}$.

Definition

If $\mathcal{B}$ and $\mathcal{C}$ are ordered bases for a vector space V with dimension n, and if U is an $n \times n$ matrix that satisfies

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Example: Consider the basis $\mathcal{B} = [\mathbf{v}_1, \mathbf{v}_2]$ where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ Then the transition matrix from $\mathcal{B}$ to $\mathcal{E} = [\mathbf{e}_1, \mathbf{e}_2]$ is
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Another example:

Let $\mathcal{B} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

We let $S = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}$ and compute its inverse:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{6 or 7 EROs}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right)$$

Thus

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Suppose $\mathcal{B} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ and $\mathcal{C} = [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3]$ are ordered bases for $\mathbb{R}^3$.

We let $S = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}$ and compute its inverse:

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To remember which order to multiply, remember that $S[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{E}}$ so S transitions from $\mathcal{B}$ to $\mathcal{E}$. To transition that from $\mathcal{E}$ to $\mathcal{C}$ we multiply it by T^{-1}.

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Here is a shortcut that removes one matrix multiplication: If we row reduce $\left(\begin{array}{c|c} T & S \end{array} \right)$ until the left side is the identity I, then the right side will be $T^{-1}S$.

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Example: to find the transition matrix from

$$\mathcal{B} = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right] \text{ to } \mathcal{C} = \left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right]$$

$$\text{Row-reduce } \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \text{ to } \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array} \right)$$

And the transition matrix from $\mathcal{B}$ to $\mathcal{C}$ is $\begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 1 \\ -1 & 0 & -1 \end{pmatrix}$

Theorem

Let $\mathcal{B} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$ *and* $\mathcal{C} = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n]$ *be bases for a vector space* V*. The transition matrix from* $\mathcal{B}$ *to* $\mathcal{C}$ *is*

$$U = \begin{pmatrix} [\mathbf{v}_1]_{\mathcal{C}} & [\mathbf{v}_2]_{\mathcal{C}} & \cdots & [\mathbf{v}_n]_{\mathcal{C}} \end{pmatrix}$$

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Three vector spaces associated with a matrix

Definition

If A is an $n \times k$ matrix then

1. The *null space of A* is the set of vectors $\mathbf{x}$ that satisfy $A\mathbf{x} = \mathbf{0}$. This is a subspace of $\mathbb{R}^k$ and consists of column vectors. We denote this $\mathcal{N}(A)$.

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3. The *row space of A* is the span of the rows of A. It is a subspace of $\mathbb{R}^k$ (interpreted as all $1 \times k$ row matrices). We don't have any special notation for this.

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Consider this matrix A and its reduced echelon form B:

$$A = \begin{pmatrix} 0 & 2 & -2 & 4 \\ 1 & 3 & -1 & 2 \\ -1 & -1 & -1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 2 & -4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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Theorem

If $L : \mathbb{R}^k \to \mathbb{R}^n$ is a linear transformation, then there is a unique $n \times k$ matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^k$.

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Define L from $\mathbb{R}^3$ to $\mathbb{R}^2$ as follows

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 + x_3 \\ x_2 - 3x_3 \end{pmatrix}$$

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And so the matrix that produces L is $A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -3 \end{pmatrix}$

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Let V be a vector space with basis $\mathcal{B}$. Let W be a vector space with basis $\mathcal{C}$. Let $L : V \to W$ be a linear transformation. Then there exists a unique matrix A that satisfies $A[\mathbf{v}]_{\mathcal{B}} = [L\mathbf{v}]_{\mathcal{C}}$.

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The method of obtaining A we saw earlier carries forward to this case:
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Example when $L : \mathbb{R}^k \to \mathbb{R}^n$

If the bases of $\mathbb{R}^k$ and $\mathbb{R}^n$ are the standard bases, the representing matrix is just the matrix we saw earlier:

$$A = \begin{pmatrix} L\mathbf{e}_1 & L\mathbf{e}_2 & \cdots & L\mathbf{e}_k \end{pmatrix}$$

If the basis in $\mathbb{R}^n$ is the standard one, then we get

$$A = \begin{pmatrix} L\mathbf{v}_1 & L\mathbf{v}_2 & \cdots & L\mathbf{v}_k \end{pmatrix}$$

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If the basis in $\mathbb{R}^n$ is $\mathcal{C} = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n]$ then we have to convert each $L\mathbf{v}_j$ into a new column vector using the transition matrix T^{-1} where $T = \left(\begin{array}{cccc} \mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_n \end{array} \right)$ One quick way to find $T^{-1}L\mathbf{v}_j$ simultaneously is to line up the the vectors as follows

$$\left(\begin{array}{cccc|cccc} \mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_n & L\mathbf{v}_1 & L\mathbf{v}_2 & \cdots & L\mathbf{v}_k \end{array} \right)$$

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and then row reduce until the left half is the identity matrix.

Example

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$$L\mathbf{x} = \begin{pmatrix} x_1 - 2x_2 + 3x_3 \\ 2x_1 + 3x_2 - 4x_3 \end{pmatrix}$$

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Find the represnting matrix for L relative to the ordered bases

$\mathcal{B} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right]$ for $\mathbb{R}^3$ and

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We need to apply L to the three basis vectors to get

$$L\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, L\mathbf{v}_2 = \begin{pmatrix} -1 \\ 5 \end{pmatrix}, L\mathbf{v}_3 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

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To get the coordinates of these relative to $\mathcal{C}$ we solve three equations.

One of those equations is:

$$x_1\mathbf{w}_1 + x_2\mathbf{w}_2 = L\mathbf{v}_1 \quad \text{or} \quad \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

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The other two apply the same proces to the vectors $L\mathbf{v}_2$ and $L\mathbf{v}_3$:

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We can solve all three at once by row reducing

$$\left(\begin{array}{cc|ccc} 1 & 1 & 2 & -1 & 1 \\ 2 & 3 & 1 & 5 & 2 \end{array}\right) \quad \text{to} \quad \left(\begin{array}{cc|ccc} 1 & 0 & 4 & -8 & 1 \\ 0 & 1 & -2 & 7 & 0 \end{array}\right)$$

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The representing matrix is just the second part: $A = \begin{pmatrix} 4 & -8 & 1 \\ -2 & 7 & 0 \end{pmatrix}$