Similar Matrices

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Theorem

Let S be an invertible $n \times n$ matrix. Suppose A and B are two $n \times n$ matrices. Then $S^{-1}ABS = (S^{-1}AS)(S^{-1}BS)$. In particular, if S is a transition matrix from a basis $\mathcal{B}$ of $\mathbb{R}^n$ to another basis $\mathcal{E}$, then the matrix representing AB with respect to $\mathcal{B}$ is the product of the representing matrices for A and B

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If $L(\mathbf{x}) = A\mathbf{x}$, find the representing matrix for A relative to the basis

$$\mathcal{B} = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right]$$

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If we find the transition matrix S from $\mathcal{B}$ to $\mathcal{E}$ (the standard basis) we need to find $S^{-1}AS$.

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$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{3 EROs}} \left(\begin{array}{ccc|rrr} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

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Therefore, The required representing matrix for L relative to $\mathcal{B}$ is

$$S^{-1}AS = \begin{pmatrix} -217 \\ 30-3 \\ 022 \end{pmatrix}$$

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If we do the same for the transformation $T\mathbf{x} = B\mathbf{x}$ where

$$B = \left(\begin{array}{lrl} 1 & -1 & 0 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{array}\right)$$

we get a representing matrix for T relative to $\mathcal{B}$:

$$S^{-1}BS = \left(\begin{array}{rrr} -1 & 0 & -1 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

Without computing $T(L(\mathbf{x})) = BA\mathbf{x}$, we can get the represention matrix for $T(L\mathbf{x})$ relative to $\mathcal{B}$ by multiplying these two representing matrices:

$$\begin{pmatrix} -1 & 0 & -1 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 & 7 \\ 3 & 0 & -3 \\ 0 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -3 & -9 \\ -4 & 2 & 14 \\ 0 & 2 & 2 \end{pmatrix}$$