Linear Transformations

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We need to find

$$T((x-1)^2) = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}, \quad T(x^2) = \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}, \quad \text{and} \quad T((x+1)^2) = \begin{pmatrix} 4 \\ 9 \\ 16 \end{pmatrix}$$

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Recall what this means: If we have a linear combination of the basis vectors (say $p(x) = -(x-1)^2 + 2x^2 - (x+1)^2$) we need only multiply the coordinates by A to get $L(p(x))$:

$$\begin{pmatrix} p(1) \\ p(2) \\ p(3) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 4 \\ 1 & 4 & 9 \\ 4 & 9 & 16 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}$$

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Matrix multiplication never involves anything except adding simple products, while evaluating a polynomial can involve rather large powers.

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Suppose we have a basis $\mathcal{B}$ for V (in the 'from' role) and a basis $\mathcal{C}$ for V (in the 'to' role). Then the representing matrix must satisfy

$$A[\mathbf{v}]_{\mathcal{B}} = [I\mathbf{v}]_{\mathcal{C}} = [\mathbf{v}]_{\mathcal{C}}$$

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That is, the transition (change of basis) matrix from $\mathcal{B}$ to $\mathcal{C}$ is the same as the representing matrix for the identity transformation.

Suppose we have 2 bases $\mathcal{B}$ and $\mathcal{E}$ in V and a basis $\mathcal{C}$ in W. Suppose A is the representing matrix for $L : V \to W$ relative to $\mathcal{E}$ and $\mathcal{C}$. What is the representing matrix for L relative to $\mathcal{B}$ and $\mathcal{C}$?

If U is the transition matrix from $\mathcal{B}$ to $\mathcal{E}$, we can just put the following definitions together

$$U[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{E}} \quad \text{and} \quad A[\mathbf{v}]_{\mathcal{E}} = [L\mathbf{v}]_{\mathcal{C}}$$

to get

$$AU[\mathbf{v}]_{\mathcal{B}} = [L\mathbf{v}]_{\mathcal{C}}.$$

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Suppose W has basis $\mathcal{D}$ in additiion to $\mathcal{C}$, and that S is the transition matrix from $\mathcal{C}$ to $\mathcal{D}$. Then we have $S[L\mathbf{v}]_{\mathcal{C}} = [L\mathbf{v}]_{\mathcal{D}}$.

If U is the transition matrix from $\mathcal{B}$ to $\mathcal{E}$, we can just put the following definitions together

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Suppose W has basis $\mathcal{D}$ in additiion to $\mathcal{C}$, and that S is the transition matrix from $\mathcal{C}$ to $\mathcal{D}$. Then we have $S[L\mathbf{v}]_{\mathcal{C}} = [L\mathbf{v}]_{\mathcal{D}}$.

Multiplying $A[\mathbf{v}]_{\mathcal{E}} = [L\mathbf{v}]_{\mathcal{C}}$ by S gives $SA[\mathbf{v}]_{\mathcal{E}} = S[L\mathbf{v}]_{\mathcal{C}} = [L\mathbf{v}]_{\mathcal{D}}$

we get that SA is the representing matrix relative to $\mathcal{E}$ and $\mathcal{D}$.

If U is the transition matrix from $\mathcal{B}$ to $\mathcal{E}$, we can just put the following definitions together

$$U[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{E}} \quad \text{and} \quad A[\mathbf{v}]_{\mathcal{E}} = [L\mathbf{v}]_{\mathcal{C}}$$

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So, we see that AU is the representing matrix for L relative to $\mathcal{B}$ and $\mathcal{C}$.

Suppose W has basis $\mathcal{D}$ in additiion to $\mathcal{C}$, and that S is the transition matrix from $\mathcal{C}$ to $\mathcal{D}$. Then we have $S[L\mathbf{v}]_{\mathcal{C}} = [L\mathbf{v}]_{\mathcal{D}}$.

Multiplying $A[\mathbf{v}]_{\mathcal{E}} = [L\mathbf{v}]_{\mathcal{C}}$ by S gives $SA[\mathbf{v}]_{\mathcal{E}} = S[L\mathbf{v}]_{\mathcal{C}} = [L\mathbf{v}]_{\mathcal{D}}$

we get that SA is the representing matrix relative to $\mathcal{E}$ and $\mathcal{D}$.

Similarly, multiplying $AU[\mathbf{v}]_{\mathcal{B}} = [L\mathbf{v}]_{\mathcal{C}}$ by S shows that SAU is the matrix representing L relative to $\mathcal{B}$ and $\mathcal{D}$.

Lets take our example from chapter 1: $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$L \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.7x_1 + 0.2x_2 \\ 0.3x_1 + 0.8x_2 \end{pmatrix} =$$

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This is a matrix transformation since $L(\mathbf{x}) = \left(\begin{array}{cc} 0.7 & 0.2 \\ 0.3 & 0.8 \end{array}\right)\mathbf{x}$. Since a vector in $\mathbb{R}^n$ is its own coordinate vector relative to the standard basis $\mathcal{E}$, we can also view this as the representing matrix relative to the bases $\mathcal{E}$ and $\mathcal{E}$.

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Now consider our example basis

$$\mathcal{B} = \left[\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right]$$

Since

$$L\mathbf{v}_1 = (1/2)\mathbf{v}_1 + 0\mathbf{v}_2 \text{ and } L\mathbf{v}_2 = 0\mathbf{v}_1 + 1\mathbf{v}_2$$

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we have

$$[L\mathbf{v}_1]_{\mathcal{B}} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \text{ and } [L\mathbf{v}_2]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and so the representing matrix for L relative to $\mathcal{B}$ and $\mathcal{B}$ is

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To complete the example, note that the transition matrix from $\mathcal{B}$ to $\mathcal{E}$ is

$$S = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$$

while the transition matrix from $\mathcal{E}$ to $\mathcal{B}$ is

$$S^{-1} = \begin{pmatrix} 3/5 & -2/5 \\ 1/5 & 1/5 \end{pmatrix}$$

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We see that

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Schematically, the representing matrix relative to $\mathcal{B}$ and $\mathcal{B}$ comes from

$$\mathcal{B} \xrightarrow{\text{transition matrix } S} \mathcal{E} \xrightarrow{\text{representing matrix } A} \mathcal{E} \xrightarrow{\text{transition matrix } S^{-1}} \mathcal{B}$$

Properties of representing matrices

Definition

We say that a linear transformation $L : V \to W$ is *invertible* if there is another linear transformation $T : W \to V$ that satisfies $T(L(\mathbf{v})) = \mathbf{v}$ and $L(T(\mathbf{w})) = \mathbf{w}$ for every $\mathbf{v} \in V$ and $\mathbf{w} \in W$. We write $T = L^{-1}$.

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By the same definition T is invertible and $L = T^{-1}$.

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If L is invertible and $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is an independent set, then so is $\{L\mathbf{v}_1, \ldots, L\mathbf{v}_m\}$. Also, if $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is a spanning set in V, then $\{L\mathbf{v}_1, \ldots, L\mathbf{v}_m\}$ is a spanning set in W.

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If L is invertible and $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is an independent set, then so is $\{L\mathbf{v}_1, \ldots, L\mathbf{v}_m\}$. Also, if $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is a spanning set in V, then $\{L\mathbf{v}_1, \ldots, L\mathbf{v}_m\}$ is a spanning set in W.

Let $\mathcal{B} = [\mathbf{v}_1, \ldots, \mathbf{v}_k]$ be a basis for V and $\mathcal{C} = [\mathbf{w}_1, \ldots, \mathbf{w}_n]$ a basis for W. Let A be the representing matrix for $L : V \to W$ relative to $\mathcal{B}$ and $\mathcal{C}$.

Then

1. The transformation $S\mathbf{x} = \sum_{j=1}^{k} x_j \mathbf{v}_j$ (from $\mathbb{R}^k$ to V), is invertible with $S(\mathcal{N}(A)) = \ker(L)$ and $S^{-1}(\ker(L)) = \mathcal{N}(A)$. Thus, the dimension of $\ker(L)$ is the nullity of A.

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2. The transformation $T\mathbf{w} = [\mathbf{w}]_{\mathcal{C}}$ (from W to $\mathbb{R}^n$) is invertible. $T(L(V))$ is the column space of A and the image of the column space by T^{-1} is $L(V)$. Thus, the dimension of $L(V)$ is the rank of A.

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Let L and T be linear transformations from V to W and let A and B be their representing matrices relative to the same bases $\mathcal{B}$ and $\mathcal{C}$. Then the representing matrix for $L + T$ is $A + B$.

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Let $L : V \to W$ and $T : W \to X$ be linear transformations and Let $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be bases for V, W, and X.

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Let $L : V \to W$ and $T : W \to X$ be linear transformations and Let $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be bases for V, W, and X. If A is the representing matrix for L with respect to $\mathcal{B}$ and $\mathcal{C}$, and B is the representing matrix for T with respect to $\mathcal{C}$ and $\mathcal{D}$, then BA is the representing matrix for TL with respect to $\mathcal{B}$ and $\mathcal{D}$.

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2. The transformation $T\mathbf{w} = [\mathbf{w}]_{\mathcal{C}}$ (from W to $\mathbb{R}^n$) is invertible. $T(L(V))$ is the column space of A and the image of the column space by T^{-1} is $L(V)$. Thus, the dimension of $L(V)$ is the rank of A.
3. The linear transformation L is invertible if and only if V and W have the same dimension and the representing matrix of L is invertible.

Let L and T be linear transformations from V to W and let A and B be their representing matrices relative to the same bases $\mathcal{B}$ and $\mathcal{C}$. Then the representing matrix for $L + T$ is $A + B$.

Let $L : V \to W$ and $T : W \to X$ be linear transformations and Let $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be bases for V, W, and X. If A is the representing matrix for L with respect to $\mathcal{B}$ and $\mathcal{C}$, and B is the representing matrix for T with respect to $\mathcal{C}$ and $\mathcal{D}$, then BA is the representing matrix for TL with respect to $\mathcal{B}$ and $\mathcal{D}$.

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So, two matrices are similar if and ony if they are representing matrices for the same linear tansformation, but different bases..