Linear Transformations

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If $L : \mathbb{R}^k \to \mathbb{R}^n$ *is a linear transformation, then there is a unique* $n \times k$ *matrix* A *such that* $L(\mathbf{x}) = A\mathbf{x}$ *for every* $\mathbf{x}$ *in* $\mathbb{R}^k$*.*

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Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k\}$ be the standard basis for $\mathbb{R}^k$. Consider the column vectors $\mathbf{a}_j = L(e_j)$ and let $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_k \end{pmatrix}$.

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If $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_k\mathbf{e}_k,$

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$$\begin{aligned} A\mathbf{x} &= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_k\mathbf{a}_k \\ &= x_1 L(\mathbf{e}_1) + x_2 L(\mathbf{e}_2) + \cdots + x_k L(\mathbf{e}_k) \\ &= L(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_k\mathbf{e}_k) = L(\mathbf{x}) \end{aligned}$$

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How do we know A is unique? Well, if $B\mathbf{x} = L(\mathbf{x})$ for every $\mathbf{x}$ in $\mathbb{R}^k$ then certainly $B\mathbf{e}_j = L(\mathbf{e}_j) = \mathbf{a}_j$.

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Example

Define L from $\mathbb{R}^3$ to $\mathbb{R}^2$ as follows $L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 + x_3 \\ x_2 - 3x_3 \end{pmatrix}$

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Then

$$L\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad L\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad L\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

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Another example: The rotation transformations R_θ satisfy

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is a linear transformation.

I claim the following are linear transformations:

$$S : \mathbb{R}^k \to V \text{ defined by } S \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} = \sum_{j=1}^{k} x_j \mathbf{v}_j$$

$$T : W \to \mathbb{R}^n \text{ defined by } T(\mathbf{w}) = [\mathbf{w}]_{\mathcal{C}}.$$

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If you accept all that, and that $T(L(S\mathbf{x}))$ is a linear transformation. then, since it goes from $\mathbb{R}^k$ to $\mathbb{R}^n$, it must be a matrix transformation. The matrix A associated to it satisfies $A[\mathbf{v}]_{\mathcal{B}} = [L\mathbf{v}]_{\mathcal{C}}$.

Let's verify that S is linear.

If $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix}$ so $\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_k + y_k \end{pmatrix}$, Let's verify that S is linear.

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$$\begin{aligned} S(\mathbf{x} + \mathbf{y}) &= \sum_{j=1}^{k} (x_j + y_j)\mathbf{v}_j \\ &= \sum_{j=1}^{k} x_j \mathbf{v}_j + \sum_{j=1}^{k} y_j \mathbf{v}_j \\ &= S(\mathbf{x}) + S(\mathbf{x}) \end{aligned}$$

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The linearity of T we already used earlier. One needs to show that

$$[\mathbf{w} + \mathbf{w}']_{\mathcal{C}} = [\mathbf{w}]_{\mathcal{C}} + [\mathbf{w}']_{\mathcal{C}} \text{ and } [\alpha\mathbf{w}]_{\mathcal{C}} = \alpha[\mathbf{w}]_{\mathcal{C}}.$$

Suppose $\mathbf{w} = \sum_{i=1}^{n} b_i \mathbf{w}_i$ and $\mathbf{w}' = \sum_{i=1}^{n} c_i \mathbf{w}_i$ so that $\mathbf{w} + \mathbf{w}' = \sum_{i=1}^{n} (b_i + c_i) \mathbf{w}_i$.

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To finish the discussion we have to verify that a composition of linear transformations is a linear transformation.

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If V, W, X are vector spaces, $L : V \to W$ is a linear transformation and $T : W \to X$ is a linear transformation, then if we define $Q : V \to X$ by $Q(\mathbf{v}) = T(L(\mathbf{v}))$, then Q is a linear transformation.

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That is, $Q(\mathbf{v}_1 + \mathbf{v}_2) = Q(\mathbf{v}_1) + Q(\mathbf{v}_2)$.

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If V, W, X are vector spaces, $L : V \to W$ is a linear transformation and $T : W \to X$ is a linear transformation, then if we define $Q : V \to X$ by $Q(\mathbf{v}) = T(L(\mathbf{v}))$, then Q is a linear transformation.

Lets verify that the addition requirement holds: let $\mathbf{v}_1$ and $\mathbf{v}_2$ be in V. Since L is linear $L(\mathbf{v}_1 + \mathbf{v}_2) = L(\mathbf{v}_1) + L(\mathbf{v}_2)$. Since T is linear $T(L(\mathbf{v}_1 + \mathbf{v}_2)) = T(L(\mathbf{v}_1) + L(\mathbf{v}_2)) = T(L(\mathbf{v}_1)) + T(L(\mathbf{v}_2))$.

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We can compose more that just 2 transformations: Because S and L (from earlier) are linear, so is $L(S(\mathbf{x}))$. Because T is linear, so is $T(L(S(\mathbf{x})))$.

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Let V be a vector space with basis $\mathcal{B}$. Let W be a vector space with basis $\mathcal{C}$. Let $L : V \to W$ be a linear transformation. Then there exists a unique matrix A that satisfies $A[\mathbf{v}]_{\mathcal{B}} = [L\mathbf{v}]_{\mathcal{C}}$.

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The previous discussion is the proof of this theorem. Let $\mathcal{B} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k]$ and $\mathcal{C} = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n]$.

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The matrix A is called the *representing matrix for L relative to the bases $\mathcal{B}$ and $\mathcal{C}$*.

The method of obtaining A we saw earlier carries forward to this case: if $\mathcal{B} = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k]$ then $A = \begin{pmatrix} [L\mathbf{v}_1]_{\mathcal{C}} & [L\mathbf{v}_2]_{\mathcal{C}} & \cdots & [L\mathbf{v}_k]_{\mathcal{C}} \end{pmatrix}$.

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Solution: First compute

$$L(\mathbf{v}_1) = \begin{pmatrix} 1 \\ 3 \end{pmatrix},\ L(\mathbf{v}_2) = \begin{pmatrix} 2 \\ 1 \end{pmatrix},\ L(\mathbf{v}_3) = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

Now compute the coordinates relative to $\mathcal{C}$ for these three vectors. We can do this with the transition matrix:

$$\begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}$$

This gives $[L\mathbf{v}_1]_{\mathcal{C}} = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, $[L\mathbf{v}_2]_{\mathcal{C}} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, and $[L\mathbf{v}_3]_{\mathcal{C}} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

So the representing matrix is $\begin{pmatrix} 5 & 0 & 3 \\ 2 & -1 & 3 \end{pmatrix}$

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Let $T : \mathcal{P}_3 \to \mathcal{P}_3$ be the linear transformation we saw earlier:
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then find the coordinates of each of those

$$[0]_{\mathcal{E}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \ [1]_{\mathcal{E}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ [2x]_{\mathcal{E}} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

Putting those columns into A gives

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Note how

$$A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ 2c \\ 0 \end{pmatrix} \quad \text{corresponds to} \quad T(a + bx + cx^2) = b + 2cx + 0x^2.$$