Linear Transformations

D. H. Luecking

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2. $L(\mathbf{v}_1 + \mathbf{v}_2) = L(\mathbf{v}_1) + L(\mathbf{v}_2)$ for every $\mathbf{v}_1, \mathbf{v}_2 \in V$.

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If A is an $n \times k$ matrix then $L(\mathbf{x}) = A\mathbf{x}$ is a linear transformation from $\mathbb{R}^k$ to $\mathbb{R}^n$. These are called *matrix transformations*.

An example from geometry

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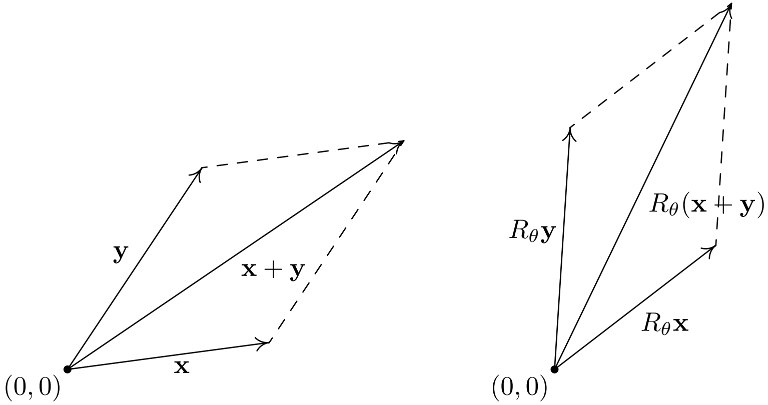

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2. $L(-\mathbf{v}) = -L(\mathbf{v})$, where $\mathbf{v}$ is any vector in V,
3. if $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are vectors in V then

$$L(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = c_1L\mathbf{v}_1 + c_2L\mathbf{v}_2 + \cdots + c_nL\mathbf{v}_n.$$

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The third property above comes from applying the two conditions in the definition repeatedly:

$$L(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots) = L(c_1\mathbf{v}_1) + L(c_2\mathbf{v}_2 + \cdots) = c_1L(\mathbf{v}_1) + L(c_2\mathbf{v}_2 + \cdots)$$

Two subspaces associated with a linear transformation

Definition

Let $L : V \to W$ be a linear transformation. The *kernel* or *null space* of L (denoted $\ker(L)$ or $\mathcal{N}(L)$) is the set

$$\ker(L) = \mathcal{N}(L) = \{\mathbf{v} \in V : L(\mathbf{v}) = \mathbf{0}_W\}.$$

Definition

Let $L : V \to W$ be a linear transformation and S a subspace of V. The *image* of S (denoted $L(S)$) is the set

$$L(S) = \{L(\mathbf{v}) : \mathbf{v} \in S\} = \{\mathbf{w} \in W : \mathbf{w} = L(\mathbf{v}) \text{ for some } \mathbf{v} \in S\}$$

The image of V, namely $L(V)$, is called the *range of L*, sometimes denoted $\mathcal{R}(L)$.

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Similarly, $\mathbf{w} = L(\mathbf{v})$ in $L(S)$ and $\alpha \in \mathbb{R}$ imply that $\alpha\mathbf{w} = \alpha L(\mathbf{v}) = L(\alpha\mathbf{v})$ is in $L(S)$.

Examples

If A is an $n \times k$ matrix and $L : \mathbb{R}^k \to \mathbb{R}^n$ is given by $L(\mathbf{x}) = A\mathbf{x}$, then $\ker(L) = \mathcal{N}(A)$.

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The range $L(\mathbb{R}^k)$ is the set of all possible $A\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^k$. Recall that if $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ are the columns of A and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}$ then

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_k\mathbf{a}_k$$

That is, the range of L is the span of the columns, that is, the column space of A. Since $\mathcal{R}(L)$ is the column space of A, the notation $\mathcal{R}(A)$ is sometimes used for the column space.

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There is nothing special about muliplying on the left, $A\mathbf{x}$. We could also regard $\mathbb{R}^n$ and $\mathbb{R}^k$ as sets of row vectors. Then we can define a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^k$ by $T(\vec{\mathbf{x}}) = \vec{\mathbf{x}}A$. In this case, the range $T(\mathbb{R}^n)$ is the row space of A.

Recall the linear transformation P_1 we defined earlier

$$P_1 \left(\begin{array}{c} x_1 \\ x_2 \end{array} \right) = \left(\begin{array}{c} x_1 \\ 0 \end{array} \right).$$

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The rotation transformations R_θ satisfy $\ker(R_\theta) = \{\mathbf{0}\}$ and $R_\theta(\mathbb{R}^2) = \mathbb{R}^2$.

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(Check that this is a linear transformation.) Its kernel and range:

$$\ker(T) = \{0\} \text{ and } T(\mathcal{P}_3) = \{a + bx + cx^2 + dx^3 \mid a = 0\}$$