Row and Column Spaces

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01 Mar 2024

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If A is an $n \times k$ matrix then

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3. The *row space of A* is the span of the rows of A. It is a subspace of $\mathbb{R}^k$ (interpreted as all $1 \times k$ row matrices). We don't have any special notation for this.

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Setting $x_3 = \alpha$ and $x_4 = \beta$ we get

$$\mathcal{N}(A) = \left\{ \left(\begin{array}{c} -2\alpha + 4\beta \\ \alpha - 2\beta \\ \alpha \\ \beta \end{array} \right) \middle| \, \alpha \in \mathbb{R} \right\} = \mathrm{Span}\left(\left(\begin{array}{r} -2 \\ 1 \\ 1 \\ 0 \end{array} \right), \left(\begin{array}{r} 4 \\ -2 \\ 0 \\ 1 \end{array} \right) \right)$$

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$$\begin{pmatrix} 0 & 2 & -2 & 4 \\ 1 & 3 & -1 & 2 \\ -1 & -1 & -1 & 2 \end{pmatrix} \xrightarrow{\text{6 EROs}} \begin{pmatrix} 1 & 0 & 2 & -4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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That means that if we reduce the matrix to echelon form the echelon form has the same row space.

For our example matrix, the matrix A and its reduced echelon form B are:

$$A = \begin{pmatrix} 0 & 2 & -2 & 4 \\ 1 & 3 & -1 & 2 \\ -1 & -1 & -1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 2 & -4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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3. A basis for the row space of A consists of the nonzero rows of B.

Note the following consequences (B is still the echelon form of A):

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The fourth is true because every column contains a leading 1 or does not.