Coordinates and Transitions

D. H. Luecking

28 Feb 2024

Another example:

Let $\mathcal{B} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ where

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$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{6 or 7 EROs}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right)$$

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This means, to find the coordinates of $\mathbf{v}$ relative to $\mathcal{B}$ we need to multiply $\mathbf{v}$ by S^{-1} for example,

$$\text{if } \mathbf{v} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} \text{ then } [\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ -12 \end{pmatrix}$$

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and this means that

$$\mathbf{v} = 4\mathbf{v}_1 + 7\mathbf{v}_2 - 12\mathbf{v}_3.$$

Now, suppose we have another basis, $\mathcal{C} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ where

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and } \mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

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We can multiply the appropriate transition matrices to get the transition matrix from (for example) $\mathcal{B}$ to $\mathcal{C}$:

Since $[\mathbf{x}]_{\mathcal{C}} = T^{-1}[\mathbf{x}]_{\mathcal{E}}$ and $[\mathbf{x}]_{\mathcal{E}} = S[\mathbf{x}]_{\mathcal{B}}$ we have $[\mathbf{x}]_{\mathcal{C}} = T^{-1}S[\mathbf{x}]_{\mathcal{B}}$.

Here's an example in $\mathcal{P}_3$. Consider the bases

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$$[x^2]_{\mathcal{E}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, [(x+1)^2]_{\mathcal{E}} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, [(x-1)^2]_{\mathcal{E}} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},$$

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I claim that the transition matrix from $\mathcal{B}$ to $\mathcal{E}$ is

$$U = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & -2 \\ 1 & 1 & 1 \end{pmatrix}$$

and the transition matrix from $\mathcal{E}$ to $\mathcal{B}$ is its inverse:

$$U^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 1/2 & 1/4 & 0 \\ 1/2 & -1/4 & 0 \end{pmatrix}$$

This means if (for example) we want $[p(x)]_{\mathcal{B}}$ for the polynomial $p(x) = 1 + 2x - 4x^2$, we have

$$= [p(x)]_{\mathcal{B}} = U^{-1}[p(x)]_{\mathcal{E}} = \begin{pmatrix} -1 & 0 & 1 \\ 1/2 & 1/4 & 0 \\ 1/2 & -1/4 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ 0 \end{pmatrix}$$

How do we verify that claim? The easiest way is to observe that the addition and scalar multiplication of vectors matches the addition and scalar multiplication of the coordinate vectors.

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$$[\alpha\mathbf{x} + \beta\mathbf{y}]_\mathcal{B} = \alpha[\mathbf{x}]_\mathcal{B} + \beta[\mathbf{y}]_\mathcal{B}$$

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$$[\mathbf{v}]_{\mathcal{E}} = [c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3]_{\mathcal{E}} = c_1[\mathbf{v}_1]_{\mathcal{E}} + c_2[\mathbf{v}_2]_{\mathcal{E}} + c_3[\mathbf{v}_3]_{\mathcal{E}}$$

$$= \begin{pmatrix} [\mathbf{v}_1]_{\mathcal{E}} & [\mathbf{v}_2]_{\mathcal{E}} & [\mathbf{v}_3]_{\mathcal{E}} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

That last equation gives us a matrix

$$U = \left(\begin{array}{ccc} [\mathbf{v}_1]_{\mathcal{E}} & [\mathbf{v}_2]_{\mathcal{E}} & [\mathbf{v}_3]_{\mathcal{E}} \end{array} \right)$$

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Theorem

Let $\mathcal{B} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$ and $\mathcal{C} = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n]$ be bases for a vector space V. The transition matrix from $\mathcal{B}$ to $\mathcal{C}$ is
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3. Equating these two versions of $[\mathbf{v}]_{\mathcal{E}}$ gives $S[\mathbf{v}]_{\mathcal{B}} = T[\mathbf{v}]_{\mathcal{C}}$, and then multiplying by S^{-1} gives $[\mathbf{v}]_{\mathcal{B}} = S^{-1}T[\mathbf{v}]_{\mathcal{C}}$, so $S^{-1}T$ transitions from $\mathcal{C}$ to $\mathcal{B}$. Inverting $S^{-1}T$ gives the transition the other way.

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Here is a quick to do this in $\mathbb{R}^n$: Take the vectors in $\mathcal{B}$ and make a matrix S. Take the vectors in $\mathcal{C}$ and make a matrix T. Put these together in a single partitioned matrix $\left(S \mid T\right)$.

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Here is a quick to do this in $\mathbb{R}^n$: Take the vectors in $\mathcal{B}$ and make a matrix S. Take the vectors in $\mathcal{C}$ and make a matrix T. Put these together in a single partitioned matrix $\left(S \mid T\right)$. Then the transition matrix $S^{-1}T$ from $\mathcal{C}$ to $\mathcal{B}$ is gotten by row reduction:

$$\left(S \mid T\right) \longrightarrow \left(I \mid S^{-1}T\right)$$