# Basis and Dimension

D. H. Luecking

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Example: Find a basis for $\mathbb{R}^4$ that contains the vectors

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Start by reducing the following matrix to echelon form, but keep track of the operations,

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & -6 \\ 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_4} \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & -6 \\ 0 & 0 \end{pmatrix} \xrightarrow{R_3 + 6R_2} \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

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Now, add two more columns that produce an invertible matrix, then perform the reverse of each of the above EROs in the opposite order:

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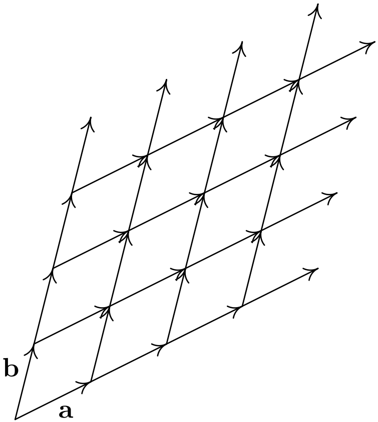
## Coordinates

Consider the following picture:




This illustrates how any point $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in $\mathbb{R}^2$ can be written as a combination $\mathbf{x} = x_1'\mathbf{a} + x_2'\mathbf{b}$.

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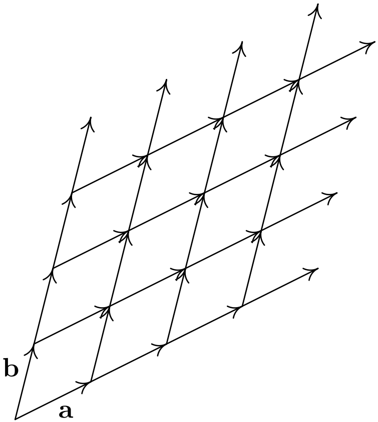


This illustrates how any point $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in $\mathbb{R}^2$ can be written as a combination $\mathbf{x} = x_1'\mathbf{a} + x_2'\mathbf{b}$. The numbers $x_1'$ and $x_2'$ are not the same as $x_1$ and $x_2$ but they are still called *coordinates relative to the basis* $\mathbf{a}, \mathbf{b}$.

Example: In our population example we had a basis for $\mathbb{R}^2$ consiting of vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

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In terms of these special coordinates, the action of $A$ is a whole lot simpler: The coordinates of $A^n\mathbf{x}$ are $(1/2)^n\alpha$ and $\beta$.

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Finding the coordinates $[\mathbf{v}]_{\mathcal{B}}$ usually requires solving some system of equations: equate $\mathbf{v}$ to the linear combination and solve for the $c_j$.

Going the other way is easy: if we know the coordinates $c_j$ and the ordered basis $\mathcal{B}$ we just use

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Given a vector $\mathbf{v}$ and an ordered basis $\mathcal{B}$, what is $[\mathbf{v}]_{\mathcal{B}}$?

Given a second basis $\mathcal{C}$, what is the relationship between $[\mathbf{v}]_{\mathcal{C}}$ and $[\mathbf{v}]_{\mathcal{B}}$?

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$$\text{if } \mathbf{v} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \quad \text{then} \quad \mathbf{v} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \cdots + c_n\mathbf{e}_n.$$

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If $\mathcal{E} = [1, x, x^2]$ in $\mathcal{P}_3$ then $[a + bx + cx^2]_{\mathcal{E}} = \left( \begin{array}{c} a \\ b \\ c \end{array} \right)$.

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If $\mathcal{E} = [E_{11}, E_{12}, E_{21}, E_{22}]$ in $\mathbb{R}^{2\times 2}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

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Suppose, in $\mathbb{R}^n$ we have another ordered basis $\mathcal{B} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$. Then finding $[\mathbf{v}]_{\mathcal{B}}$ amounts to solving

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{v} \implies [\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

The left side of this is the same as

$$\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{v}$$

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By the properties of a basis, this system has a solution for every choice of $\mathbf{v}$, which means this matrix

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So if we find the inverse $S^{-1}$ we can multiply it times $S[\mathbf{v}]_{\mathcal{B}} = \mathbf{v}$ to get $[\mathbf{v}]_{\mathcal{B}} = S^{-1}\mathbf{v}$.

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We call $S^{-1}$ the *transition matrix* from $\mathcal{E}$ to $\mathcal{B}$. Another trem used is *change of basis matrix*.

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If $\mathcal{B}$ and $\mathcal{C}$ are ordered bases for a vector space $V$ with dimension $n$, and if $U$ is an $n \times n$ matrix that satisfies

$$U[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{C}}$$

then we call $U$ the *transition or change of basis matrix* from $\mathcal{B}$ to $\mathcal{C}$.

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Time for an example: Consider the basis $\mathcal{B} = [\mathbf{v}_1, \mathbf{v}_2]$ where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ Then the transition matrix from $\mathcal{B}$ to $\mathcal{E} = [\mathbf{e}_1, \mathbf{e}_2]$ is $S = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$

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$$U[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{C}}$$

then we call $U$ the *transition or change of basis matrix* from $\mathcal{B}$ to $\mathcal{C}$.

The standard basis $\mathcal{E}$ in $\mathbb{R}^n$ satisfies $\mathbf{v} = [\mathbf{v}]_{\mathcal{E}}$. From the equations we obtained earlier: $S[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{E}}$ and $[\mathbf{v}]_{\mathcal{B}} = S^{-1}[\mathbf{v}]_{\mathcal{E}}$ we see that $S$ is the transition matrix from $\mathcal{B}$ to $\mathcal{E}$ while $S^{-1}$ is the transition matrix from $\mathcal{E}$ to $\mathcal{B}$.

Time for an example: Consider the basis $\mathcal{B} = [\mathbf{v}_1, \mathbf{v}_2]$ where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ Then the transition matrix from $\mathcal{B}$ to $\mathcal{E} = [\mathbf{e}_1, \mathbf{e}_2]$ is $S = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ and the transition matrix from $\mathcal{E}$ to $\mathcal{B}$ is $S^{-1} = \begin{pmatrix} 3/5 & -2/5 \\ 1/5 & 1/5 \end{pmatrix}$.

Thus, if $\mathbf{v} = \begin{pmatrix} 8000 \\ 2000 \end{pmatrix}$ then $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ where

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = [\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} 3/5 & -2/5 \\ 1/5 & 1/5 \end{pmatrix} \begin{pmatrix} 8000 \\ 2000 \end{pmatrix} = \begin{pmatrix} 4000 \\ 2000 \end{pmatrix}$$