Basis and Dimension

D. H. Luecking

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Then, after subtraction, we would have

$$\mathbf{0} = (\alpha_1 - \beta_1)\mathbf{v}_1 + (\alpha_2 - \beta_2)\mathbf{v}_2 + \cdots + (\alpha_n - \beta_n)\mathbf{v}_n$$

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Because $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is independent, this can only happen if

$$\alpha_1 - \beta_1 = 0, \quad \alpha_2 - \beta_2 = 0, \ \ldots, \ \alpha_n - \beta_n = 0$$

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It is routine to verify that this is a basis for $\mathbb{R}^2$:

$$\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \xrightarrow{R_2+R_1} \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix} \xrightarrow{(1/5)R_2} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

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So the set is both independent and spanning.

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If we do this with our population vector:

$$\begin{pmatrix} 8000 \\ 2000 \end{pmatrix} = 4000 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2000 \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

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So, we get

$$\begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}^n \begin{pmatrix} 8000 \\ 2000 \end{pmatrix} = (1/2)^n 4000 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2000 \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

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If we put these in a matrix, it is already in echelon form and clearly satisfies the tests for spanning and independence.

The standard basis for $\mathcal{P}_n$ is $\{1, x, \ldots, x^{n-1}\}$. In fact, the definition of a polynomial is that it is a linear combination of powers of x.

The standard basis for $\mathbb{R}^{n \times k}$ consists of all $\{E_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$, where E_{ij} is the matrix with a 1 in position ij and zeros everywhere else.

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If a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V is spanning, then any set of vector with more than n elements must be dependent.

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Let's see why the first statement is true. Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a set in V with $k > n$.

Since the set of $\mathbf{v}_i$ is spanning, there are linear combinations that produce each $\mathbf{u}_j$:

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If we can make each term in parentheses equal to zero for some *nontrivial* set of α_j, then we will have shown that the set of $\mathbf{u}_j$ is dependent.

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But this system has more variables (α_j, $1 \le j \le k$) than equations, so it has non trivial solutions.

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Why do bases have to be the same size? If a $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V, suppose $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k\}$ is another basis.

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Definition

The dimension of a vector space is the number of elements in any basis.

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The set of upper triangular, 2×2 matrices has dimension 3. A basis is

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

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Theorem

If the dimension of V is n then any set in V with more than n elements is dependent, and any set with fewer than n elements is not spanning.

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The set of upper triangular, 2×2 matrices has dimension 3. A basis is

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

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$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k + \alpha_{k+1} \mathbf{w} = \mathbf{0}$$

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Example:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Put these columns in a matrix and reduce to echelon form:

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & -1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{5 EROs}} \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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But it also tells us that if we take a linear combination and equate it to $\mathbf{0}$:

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In general, if you keep the vectors corresponding to columns with leading 1s, (and discard the rest) you get an independent set with the same span.