Spans and Independence

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Example: Are the following vectors in $\mathbb{R}^4$ spanning? Are they independent?

$$\begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 1 \\ 1 \end{pmatrix},$$

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$$\begin{pmatrix} 2 & 2 & 4 & -2 \\ 1 & 3 & 0 & 2 \\ 3 & -2 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{(1/2)R_1} \begin{pmatrix} 1 & 1 & 2 & -1 \\ 1 & 3 & 0 & 2 \\ 3 & -2 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix}$$

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$$\xrightarrow{\text{3 type III}} \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 2 & -2 & 3 \\ 0 & -5 & -5 & 4 \\ 0 & -2 & -3 & 3 \end{pmatrix} \xrightarrow{(1/2)R_2} \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -1 & 3/2 \\ 0 & -5 & -5 & 4 \\ 0 & -2 & -3 & 3 \end{pmatrix}$$

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$$\xrightarrow{\text{2 type III}} \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -1 & 3/2 \\ 0 & 0 & -10 & 23/2 \\ 0 & 0 & -5 & 6 \end{pmatrix} \xrightarrow{\text{3 EROs}} \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -1 & 3/2 \\ 0 & 0 & 1 & -23/20 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Problem: Let V be the collection of all 2×2 upper triangular matrices, that is, matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. Are the following matrices in V spanning? Are they independent?

$$A_1 = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}, A_2 = \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & -1 \\ 0 & -3 \end{pmatrix}$$

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Solution: Consider the equation

$$\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 = B$$

Where B is any matrix in V. Computing the left hand side we get

$$\begin{pmatrix} \alpha_1 + 2\alpha_2 & 3\alpha_1 + 5\alpha_2 - \alpha_3 \\ 0 & 2\alpha_1 + \alpha_2 - 3\alpha_3 \end{pmatrix} = B$$

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In either case we get 3 equations in the unknowns α_j:

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We have to bring the system matrix into echelon form:

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 5 & -1 \\ 2 & 1 & -3 \end{pmatrix} \xrightarrow{\text{4 EROs}} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

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The row of zeros means that A_1, A_2, A_3 is not a spanning set. There is no leading 1 in column 3, so this is not an independent set.

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The lack of a leading 1 in column 3 means that if $a = b = c = 0$, the system of equations will have a nontrivial solution: since α_3 is a free variable, we can set it equal to any nonzero value.

An example in $\mathcal{P}_3$

Are the following ellements of $\mathcal{P}_3$ independent? Are they spanning?

$$p_1(x) = x^2 - 2x + 1,\ p_2(x) = x - 1,\ p_3(x) = 1,\ p_4(x) = x^2 + x$$

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$$(\alpha_1 + \alpha_4)x^2 + (-2\alpha_1 + \alpha_2 + \alpha_4)x + (\alpha_1 - \alpha_2 + \alpha_3) = ax^2 + bx + c$$

Leading to a system of equations for the α_j.

$$\begin{aligned} \alpha_1 + \qquad\qquad\quad + \alpha_4 &= a \\ -2\alpha_1 + \alpha_2 + \qquad + \alpha_4 &= b \\ \alpha_1 - \alpha_2 + \alpha_3 \qquad\quad &= c \end{aligned}$$

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The system matrix and its echelon form is:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ -2 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{pmatrix} \xrightarrow{\text{3 EROs}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

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This set of polynomials spans $\mathcal{P}_3$ but is not independent.

Note that sometimes the matrix we need to process is square, sometimes (as in this case it is not). When it is square, say it is $n \times n$. Then either the relevant vectors are both independent and spanning or they are neither.

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The vectors are both independent and spanning if and only if that square matrix is invertible. Otherwise they are neither.

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If the matrix is not square, there are a couple of things we can always conclude just from its size. If the matrix is wider than it is high, the vectors cannot be independent. If the matrix is higher than it is wide, the vectors cannot be spanning.