Spans and Independence

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Definition

In $\mathbb{R}^n$, for $1 \leq j \leq n$, we let $\mathbf{e}_j$ be the vector that has a 1 in position j and zeros in every other position.

In $\mathbb{R}^3$, the set $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is spanning. To see this

$$\alpha_2 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 = \alpha_2 \left(\begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) + \alpha_2 \left(\begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) + \alpha_3 \left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) = \left(\begin{array}{c} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{array} \right)$$

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We will consider two problems connected with spanning:

1. Given a set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in V, how can we determine whether it is spanning?
2. If it is not spanning, how can we decide if some vector $\mathbf{w}$ belongs to the span?

Actually, the second of these is in some ways the easiest:
Example:

$$\text{Is} \quad \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \quad \text{in the span of} \quad \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix}?$$

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Answering this is equivalent to the following: Are there any scalars x_1, x_2, x_3 such that

$$x_1 \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

The equation on the last slide is the same as

$$\left(\begin{array}{r} x_1 + 2x_2 - x_3 \\ 4x_1 - x_2 + 5x_3 \\ 3x_2 - 3x_3 \end{array}\right) = \left(\begin{array}{r} 2 \\ -1 \\ 5 \end{array}\right)$$

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Equating corresponding positions we get a system of equations whose augmented matrix is

$$\left(\begin{array}{rrr|r} 1 & 2 & -1 & 2 \\ 4 & -1 & 5 & -1 \\ 0 & 3 & -3 & 5 \end{array}\right) \xrightarrow{\text{3 EROs}} \left(\begin{array}{ccr|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

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The last row tells us that there are no solutions and so $(2, -1, 5)^T$ is not in the span of those three vectors.

One might now ask exactly what vectors belong to the span of our example set? That can be answered by putting unknowns on the right sides and trying to solve it anyway:

$$\left(\begin{array}{rrr|c} 1 & 2 & -1 & b_1 \\ 4 & -1 & 5 & b_2 \\ 0 & 3 & -3 & b_3 \end{array}\right) \xrightarrow{R_2-4R_1} \left(\begin{array}{rrr|c} 1 & 2 & -1 & b_1 \\ 0 & -9 & 9 & b_2-4b_1 \\ 0 & 3 & -3 & b_3 \end{array}\right)$$

$$\xrightarrow{(-1/9)R_2} \left(\begin{array}{rrr|c} 1 & 2 & -1 & b_1 \\ 0 & 1 & -1 & -b_2/9+4b_1/9 \\ 0 & 3 & -3 & b_3 \end{array}\right)$$

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So $(b_1, b_2, b_3)^T$ is in the span if and only if $b_3 + b_2/3 - 4b_1/3 = 0$.

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So $(b_1, b_2, b_3)^T$ is in the span if and only if $b_3 + b_2/3 - 4b_1/3 = 0$. That means that some vectors are not in the span and so the set is not spanning.

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Example 1: Do the following vectors span $\mathbb{R}^3$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix}$$

Solution: a linear combination of these equated to any vector produces the following system matrix:

$$\left(\begin{array}{rrrr} 1 & 1 & 2 & 0 \\ 2 & -3 & -1 & 5 \\ 3 & 2 & 5 & 1 \end{array}\right) \xrightarrow[R_3-3R_1]{R_2-2R_1} \left(\begin{array}{rrrr} 1 & 1 & 2 & 0 \\ 0 & -5 & -5 & 5 \\ 0 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{-(1/5)R_2} \left(\begin{array}{rrrr} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{array}\right) \xrightarrow{R_3+R_2} \left(\begin{array}{rrrr} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

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They do not span $\mathbb{R}^3$.

Example 2: Do the following matrices span $\mathbb{R}^{2\times 2}$?

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Consider the equation

$$x_1 \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\text{Or} \quad \begin{pmatrix} x_1 + x_3 & 2x_1 + 2x_2 - x_3 \\ -x_2 - x_3 & x_1 + x_2 + x_3 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

That last equation boils down to 4 equations with the following system matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{7 EROs}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

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Independence

Lets go back to this example:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix}.$$

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When one or more vectors in a set are linear combinations of the other vectors, we say the set is linearly dependent. Working out which vector depends on the others can be tricky and is often not necessary. Therefore, we adopt the following definition.

Definition

A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is said to be *linearly dependent* if there exists scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, not all of which are zero, such that

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In our earlier example we had $\mathbf{v}_3 = \mathbf{v}_1 - \mathbf{v}_2$ so that $\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$. This is a nontrivial linear combination equal to $\mathbf{0}$ and so the vectors are dependent.

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Determining whether a set of vectors is independent or dependent again amounts to a system of equations. But this time, it is not whether the system has a solution, but whether it has a nontrivial solution.

Consider:

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n = \mathbf{0}$$

If we think of the α_j as variables, we are asking if this system of equations has a solution other than $\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_n = 0$.

Example:
Are the following vectors independent?

$$\mathbf{w}_1 = \begin{pmatrix} 1 & 3 & 5 \end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix}, \mathbf{w}_3 = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

Here the equation

$$\alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2 + \alpha_3 \mathbf{w}_3 = \mathbf{0}$$

becomes the system

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= 0 \\ 3\alpha_1 \phantom{{}+2\alpha_2} + 2\alpha_3 &= 0 \\ 5\alpha_1 + 2\alpha_2 + 3\alpha_3 &= 0 \end{aligned}$$

The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 0 & 2 & 0 \\ 5 & 2 & 3 & 0 \end{array}\right) \xrightarrow[R_3-5R_1]{R_2-3R_1}$$

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And we row reduce it:

$$\left(\begin{array}{rrr|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & -3 & -2 & 0 \end{array}\right) \xrightarrow{(-1/3)R_2} \left(\begin{array}{rrr|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & -3 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{R_3+3R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right) \xrightarrow{(-1)R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

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Conclusion: The only solution is the trivial one $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$, so the original set of vectors is independent.

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4. If, in that echelon form, there is a column without a leading 1, the set of vectors is dependent, otherwise it is independent.