Subspaces

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16 Feb 2024

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We'll concentrate on the known vector spaces $\mathbb{R}^{n\times k}$ and $\mathcal{P}_n$ and show how to determine whether a given subset is a subspace.

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This is not a subspace of $\mathbb{R}^2$ because C1 fails: $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is in W_1 but $-2\mathbf{x} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$ is not in W_1.

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Note: W_1 does satisfy C2, but that alone isn't good enough.

Another example

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Note that this was almost exactly the same as verifying that

$$W_4 = \left\{ \left(\begin{array}{c} a \\ b \\ c \end{array} \right) : a = b + c \right\}$$

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Here's a couple of examples in $\mathbb{R}^{n\times n}$:

$$W_5 = \{A \in \mathbb{R}^{n\times n} : A^T = A\} \text{ and } W_6 = \{A \in \mathbb{R}^{n\times n} : A^T = -A\}$$

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Definition

If A is an $n \times k$ matrix then the set of vectors $\mathbf{x} \in \mathbb{R}^k$ that satisfy $A\mathbf{x} = \mathbf{0}$ is called the *null space of A* and we denote it by $\mathcal{N}(A)$

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Example: Find the null space of $A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 1 & 3 & 5 & 4 \\ 2 & 5 & 8 & 4 \end{pmatrix}$

Solution: A is the system matrix of $A\mathbf{x} = \mathbf{0}$ so we solve that system with an augmented matrix: add a column of 0s to A:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 1 & 3 & 5 & 4 & 0 \\ 2 & 5 & 8 & 4 & 0 \end{array}\right) \xrightarrow{4 \text{ ERO's}} \left(\begin{array}{cccc|c} 1 & 0 & -1 & -8 & 0 \\ 0 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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That gives

$$\begin{aligned} x_1 &= \quad x_3 + 8x_4 \\ x_2 &= -2x_3 - 4x_4 \end{aligned}$$

and so

$$\mathcal{N}(A) = \left\{ \left(\begin{array}{c} \alpha + 8\beta \\ -2\alpha - 4\beta \\ \alpha \\ \beta \end{array} \right) : \alpha, \beta \in \mathbb{R} \right\}$$

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For example, suppose we want to create a subspace of $\mathbb{R}^3$ that contains the two vectors $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$.

Such a subspace would have to contain $\alpha\mathbf{a}$ for every α in $\mathbb{R}$ as well as $\beta\mathbf{b}$ for every β in $\mathbb{R}$, as well as every sum of such vectors.

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Such a subspace would have to contain $\alpha\mathbf{a}$ for every α in $\mathbb{R}$ as well as $\beta\mathbf{b}$ for every β in $\mathbb{R}$, as well as every sum of such vectors. If we let W be the set of all these sums:

$$W = \{\alpha\mathbf{a} + \beta\mathbf{b} : \alpha, \beta \in \mathbb{R}\}$$

then W is a subspace of $\mathbb{R}^3$.

If we note that

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Definition

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are vectors in a vector space V, a sum of the form

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars, is called a *linear combination of* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. The set of all such linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is called the *span of* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

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The previous example, where we wanted a subspace of $\mathbb{R}^3$ that contained

$$\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$$

was

$$W = \mathrm{Span}(\mathbf{a}, \mathbf{b}) = \mathrm{Span}\left(\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \right)$$

Also, the null space we calculated earlier consisted of all vectors of the form

$$\begin{pmatrix} \alpha + 8\beta \\ -2\alpha - 4\beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 8 \\ -4 \\ 0 \\ 1 \end{pmatrix}$$

This shows that $\mathcal{N}(A)$ is the span of the two column vectors above.