Vectors

D. H. Luecking

14 Feb 2024

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There is a third way.

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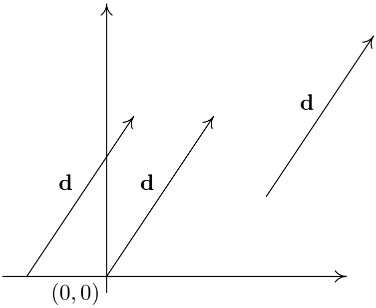

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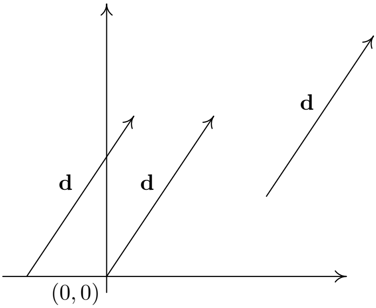

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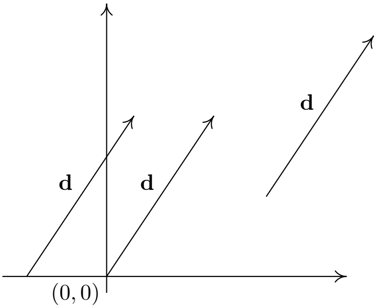

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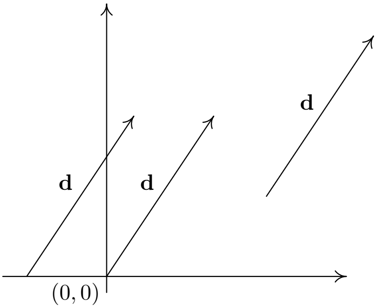

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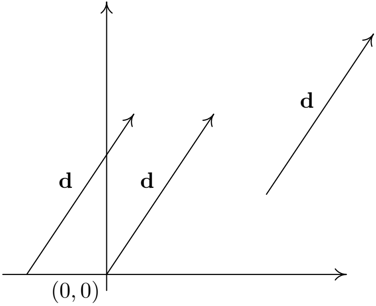

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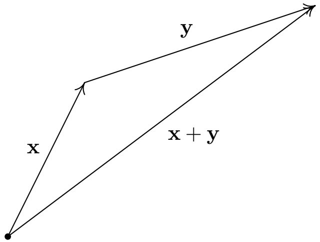

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- There are well-define ways to convert from one to to another.
- They all satisfy a common set of properties.

Vector spaces

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C2 If $a = -b$ and $c = -d$ is $(a + c, b + d)$ in V?

It can be proved (solely from the axioms) that $0\mathbf{x} = \mathbf{0}$ and $(-1)\mathbf{x} = -\mathbf{x}$, it must be that A3 and A4 follow from C1 (provided V is a subset of a known vector space).

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Hidden in the definitions is the means to solve problems like $\mathbf{x} + \mathbf{a} = \mathbf{b}$ (obtaining $\mathbf{x} = \mathbf{b} - \mathbf{a}$) and like $\alpha\mathbf{x} = \mathbf{b}$ ($\mathbf{x} = \frac{1}{\alpha}\mathbf{b}$, provided $\alpha \neq 0$).

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Conditions C1 and C2 are part of the definition of scalar multiplication and addition for matrices.

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These are both sort of obvious.

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It is a subset of $\mathbb{R}^k$, so we only have to show C1 and C2. Now the system can be written $A\mathbf{x} = \mathbf{0}$, so we only have to be able to answer 'yes' to:
C1: If $A\mathbf{x} = \mathbf{0}$ and $\alpha \in \mathbb{R}$, is $A(\alpha\mathbf{x}) = \mathbf{0}$?
C2: If $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$, is $A(\mathbf{x} + \mathbf{y}) = 0$?
Both are true because $A(\alpha\mathbf{x}) = \alpha(A\mathbf{x}) = \alpha\mathbf{0} = \mathbf{0}$, and $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$.

The set $\{\mathbf{0}\}$, in any vector space, is a vector space. To prove this we would have to show that $\alpha\mathbf{0} = \mathbf{0}$. This seems obvious, but since it is stated for all vector spaces, and it is not one of the conditions, it needs a proof:

$$\alpha\mathbf{0} = \alpha\mathbf{0} + \alpha\mathbf{0} - \alpha\mathbf{0} = \alpha(\mathbf{0} + \mathbf{0}) - \alpha\mathbf{0} = \alpha\mathbf{0} - \alpha\mathbf{0} = \mathbf{0}$$