Chapter 1&2 Review

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System of linear equations (aka "system"):

Example: $$\begin{cases} \quad\quad\; 3x_2 - 3x_3 + 3x_4 = 0 \\ x_1 + 2x_2 \quad\quad\;\; + \;\; x_4 = 1 \\ 3x_1 + 6x_2 \quad\quad\; + 5x_4 = -3 \end{cases}$$

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Augmented matrix:

Example: $$\left(\begin{array}{cccc|c} 0 & 3 & -3 & 3 & 0 \\ 1 & 2 & 0 & 1 & 1 \\ 3 & 6 & 0 & 5 & -3 \end{array}\right)$$

Gauss-Jordan reduction:

$$\left(\begin{array}{cccc|c} 0 & 3 & -3 & 3 & 0 \\ 1 & 2 & 0 & 1 & 1 \\ 3 & 6 & 0 & 5 & -3 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 3 & -3 & 3 & 0 \\ 3 & 6 & 0 & 5 & -3 \end{array}\right)$$

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$$\xrightarrow{R_3 - 3R_1} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 3 & -3 & 3 & 0 \\ 0 & 0 & 0 & 2 & -6 \end{array}\right) \xrightarrow[(1/2)R_3]{(1/3)R_2} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right)$$

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We now know there is at least one solution: because there is no row with all zeros left of the bar and nonzero to the right.

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We now know there is at least one solution: because there is no row with all zeros left of the bar and nonzero to the right. The leading variables are x_1, x_2 and x_4. Since we have a free variable x_3, there are infinitely many solutions.

Echelon form and reduced echelon form:

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right) \xrightarrow[R_1 - R_3]{R_2 - R_3} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 4 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right)$$

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Number of solutions: This system has at least one solution because the echelon form does not have a row of zeros in the system part with a nonzero number in the augmented part.

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It has infinitely many solutions because the free variable x_3 can be given any value.

Solutions:

$$\left.\begin{aligned} x_1 &= -2 - 2x_3 \\ x_2 &= 3 + x_3 \\ x_4 &= -3 \end{aligned}\right\} \xrightarrow{\text{and so,}} \begin{cases} x_3 = \alpha \\ x_1 = -2 - 2\alpha \\ x_2 = 3 + \alpha \\ x_4 = -3 \end{cases}$$

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Whence the solutions are

$$(-2 - 2\alpha, 3 + \alpha, \alpha, -3) \quad \text{or} \quad \begin{pmatrix} -2 - 2\alpha \\ 3 + \alpha \\ \alpha \\ -3 \end{pmatrix}$$

Arithmetic with matrices.

Scalar multiplication: if $A = (a_{ij})$ then $\alpha A = (\alpha a_{ij})$.

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Addition: If $A = (a_{ij})_{n\times k}$ and $B = (b_{ij})_{n\times k}$ then $A + B = (a_{ij} + b_{ij})_{n\times k}$.

$$\text{Example: } \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 5 & -2 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 1 & 0 \end{pmatrix}$$

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Matrix multiplication: If $A = (a_{ij})_{n\times m}$ and $B = (b_{jk})_{m\times p}$ then $AB = C$ where $(c_{ik})_{n\times p} = \left(\sum_{j=1}^{m} a_{ij}b_{jk}\right)_{n\times p}$.

Alternatively:

$$\text{If} \quad A = \begin{pmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_n \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{pmatrix}$$

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then $AB = C$ where $(c_{ik})_{n\times p} = (\vec{\mathbf{a}}_i \mathbf{b}_k)_{n\times p}$.

$$\text{Example: } \begin{pmatrix} 1 & 3 & -2 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 6 \\ 0 & -2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & -6 \\ -1 & -1 \end{pmatrix}$$

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We can always multiply A and A^T in either order:

$$AA^T = \begin{pmatrix} 6 & -6 \\ -6 & 34 \end{pmatrix} \quad \text{and} \quad A^TA = \begin{pmatrix} 10 & 8 & -10 \\ 8 & 10 & -14 \\ -10 & -14 & 20 \end{pmatrix}$$

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For square A we can add (and subtract) A and A^T: If $A = \begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix}$ then $A + A^T = \begin{pmatrix} 6 & 3 \\ 3 & 4 \end{pmatrix}$ and $A - A^T = \begin{pmatrix} 0 & 5 \\ -5 & 0 \end{pmatrix}$.

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Any invertible matrix A is a product of elementary matrices and A^{-1} is the product of their inverses in the opposite order.

We can find the inverse of A (or perhaps discover it has none) by reducing the matrix $(A \mid I)$ to reduced echelon form.

Example: A is the matrix on the left below.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 5 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Example: A is the matrix on the left below.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 5 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 - R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -2 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

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$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 5 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2-2R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1-R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1-2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -2 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

so

$$A^{-1} = \left(\begin{array}{ccc} 5 & -2 & -1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

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Example: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & -1 \end{pmatrix} \left(\begin{array}{cc|c} 2 & 5 & 1 \\ 1 & 0 & -1 \end{array} \right) = \left(\begin{array}{cc|c} 4 & 5 & -1 \\ 10 & 15 & -1 \\ -1 & 0 & 1 \end{array} \right)$

2. $\left(\frac{A_1}{A_2}\right) B = \left(\frac{A_1 B}{A_2 B}\right)$

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3. $\left(\begin{array}{c|c} A_1 & A_2 \end{array}\right) \left(\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right) = A_1 B_1 + A_2 B_2.$

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$$\left(\begin{array}{c|c} 1 & 2 \\ 3 & 4 \end{array}\right) \left(\begin{array}{cc} 2 & 3 \\ \hline -1 & 3 \end{array}\right) = \left(\begin{array}{cc} 2 & 3 \\ 6 & 9 \end{array}\right) + \left(\begin{array}{cc} -2 & 6 \\ -4 & 12 \end{array}\right) = \left(\begin{array}{cc} 0 & 9 \\ 2 & 21 \end{array}\right).$$

Finally,

4. $$\left(\begin{array}{c} A_1 \\ \hline A_2 \end{array}\right) \left(\begin{array}{c|c} B_1 & B_2 \end{array}\right) = \left(\begin{array}{c|c} A_1B_1 & A_1B_2 \\ \hline A_2B_1 & A_2B_2 \end{array}\right)$$

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In general, if A is $n \times n$ pick any row of A (say row number i) and multiply each element in that row by the cofactor of its position:

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$

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$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$

Alternatively, do this for any column of A:

$$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$$

Examples:

$$\begin{vmatrix} 2 & -3 \\ 5 & -2 \end{vmatrix} = -4 - (-15) = 11$$

$$\begin{vmatrix} 2 & -3 & 1 \\ 5 & -2 & 2 \\ 0 & 2 & 0 \end{vmatrix} = -2 \begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix} = -2(4-5) = 2.$$

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$$\begin{vmatrix} 0 & 2 & -3 \\ 0 & 5 & -2 \\ 3 & 5 & 9 \end{vmatrix} = 3\begin{vmatrix} 2 & -3 \\ 5 & -2 \end{vmatrix} = -3(11) = -33.$$

Special cases:

Triangular matrices:

$$\begin{vmatrix} 2 & 7 & 19 \\ 0 & 3 & 21 \\ 0 & 0 & 4 \end{vmatrix} = (2)(3)(4) = 24.$$

"Block tringular":

$$\begin{vmatrix} 1 & 3 & 19 & 3 & 5 \\ 2 & 7 & 14 & -1 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 4 & 2 & 5 \\ 0 & 0 & 2 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \begin{vmatrix} 1 & 0 & 3 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{vmatrix} = (7-6)(2)(3-6) = -6$$

EROs:

$$\begin{vmatrix} 0 & 2 & 4 & 5 \\ 1 & 3 & 5 & 6 \\ 2 & 6 & 12 & -2 \\ 1 & 3 & 0 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 5 & 6 \\ 0 & 2 & 4 & 5 \\ 2 & 6 & 12 & -2 \\ 1 & 3 & 0 & 3 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 3 & 5 & 6 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 2 & -14 \\ 0 & 0 & -5 & -3 \end{vmatrix} = -2 \begin{vmatrix} 1 & 3 & 5 & 6 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & -5 & -3 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 1 & 3 & 5 & 6 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & -38 \end{vmatrix} = (-2)(-76) = 152.$$