Cramer's Rule

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Recall that an $n \times n$ system can be represented as $A\mathbf{x} = \mathbf{b}$ where A is and $n \times n$ matrix, $\mathbf{x}$ is the column vector of variables $x_1, x_2, \ldots, x_n$ and $\mathbf{b}$ is the column vector of the right-hand sides of the equation.

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If A is an invertible matrix we can multiply $A\mathbf{x} = \mathbf{b}$ on both sides by the inverse of A. This gives:

$$\mathbf{x} = A^{-1}\mathbf{b} = \left(\frac{1}{\det(A)}\,\mathrm{adj}\,A\right)\mathbf{b}$$

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If we compare the jth position on both sides we get

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The expression in parentheses looks like a determinant. In fact, it is exactly the determinant of the matrix obtained from A by replacing its jth column with $\mathbf{b}$.

Theorem

Let A be an invertible $n \times n$ matrix and $\mathbf{b}$ any $n \times 1$ column vector. Let B_j be the matrix obtained from A by replacing its jth column with $\mathbf{b}$. Then the solution of $A\mathbf{x} = \mathbf{b}$ is given by

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Example: for the following system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 2x_1 + 2x_2 + x_3 &= 0 \\ x_1 + 2x_2 + 3x_3 &= 0 \end{aligned}$$

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We compute

$$\det(A) = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -4, \quad \det(B_1) = \begin{vmatrix} 2 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{vmatrix} = 8$$

Continuing:

$$\det(B_2) = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 0 & 3 \end{vmatrix} = -10, \quad \det(B_3) = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 2 & 0 \\ 1 & 2 & 0 \end{vmatrix} = 4$$

Therefore

$$x_1 = \frac{8}{-4} = -2, \quad x_2 = \frac{-10}{-4} = 5/2, \quad x_3 = \frac{4}{-4} = -1.$$