More Determinants

D. H. Luecking

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We can easily find the determinant of an elementary matrix without any computations.

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If E is an elementary matrix, then $\det(EB) = \det(E)\det(B)$. We can repeat this to get (for example)

$$\det(E_1E_2B) = \det(E_1)\det(E_2B) = \det(E_1)\det(E_2)\det(B).$$

And keep repeating it to get

$$\det(E_1 E_2 \cdots E_k B) = \det(E_1)\det(E_2)\cdots\det(E_k)\det(B).$$

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If A is invertible then $A = E_1 E_2 \cdots E_k$ for some elementary matrices E_i. Applying our previous formula with $B = I$ we get

$$\det(A) = \det(E_1)\det(E_2)\cdots\det(E_k)\det(I)$$

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Corollary

If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

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If $i \neq j$ then that expression is the determinant of the matrix whose jth row has been set equal to its ith row. But such a determinant is 0. So, the product $A \operatorname{adj} A$ has zeros everywhere off the main diagonal.

As a consequence, we have a formula for the inverse of a matrix:

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj} A$$

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For 2×2 matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & -ab + ab \\ cd - cd & -bc + ad \end{pmatrix} = (ad - bc)I$$

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and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{d}{ad - bc} & -\dfrac{b}{ad - bc} \\ -\dfrac{c}{ad - bc} & \dfrac{a}{ad - bc} \end{pmatrix}$$

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \frac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

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And so,

$$A^{-1} = -\frac{1}{9} \begin{pmatrix} 1 & -13 & -5 \\ -2 & -1 & 1 \\ -2 & 8 & 1 \end{pmatrix}$$

Some examples of computing determinants.

$$A = \begin{pmatrix} 2 & 5 & -1 & 3 \\ 1 & 2 & 3 & 4 \\ 0 & 4 & 1 & 2 \\ 2 & 4 & 6 & 8 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 5 & 2 & -3 \\ 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 6 & 8 \end{pmatrix}$$

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$\det(B) = \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} \cdot \begin{vmatrix} 1 & 2 \\ 6 & 8 \end{vmatrix} = (4-5)(8-12) = 4$

Some more examples:

$$C = \begin{pmatrix} 0 & 5 & 0 & 4 \\ 3 & -8 & 0 & 4 \\ -2 & 5 & 3 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 5 & 3 & 7 \\ 1 & 6 & -2 & 4 \\ 2 & 10 & 1 & 3 \\ 1 & 5 & 4 & 0 \end{pmatrix}$$

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$$\det C = 3 \begin{vmatrix} 0 & 5 & 4 \\ 3 & -8 & 4 \\ 0 & 1 & -1 \end{vmatrix} = 3(3)(-1) \begin{vmatrix} 5 & 4 \\ 1 & -1 \end{vmatrix} = -9(-5-4) = 81$$

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After 3 EROs the determinant of matrix D is the same as

$$\begin{vmatrix} 1 & 5 & 3 & 7 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & -5 & -11 \\ 0 & 0 & 1 & -7 \end{vmatrix} = \begin{vmatrix} 1 & 5 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} -5 & -11 \\ 1 & -7 \end{vmatrix} = (1-0)(35+11) = 46$$