Determinants

D. H. Luecking

05/07 Feb 2024

If we have reason to believe an $n \times n$ system has a unique solution, is there a formula for that solution?

If we have reason to believe an $n \times n$ system has a unique solution, is there a formula for that solution? Here is a what happens when $n = 1$:

$$ax_1 = b, \quad x_1 = \frac{b}{a}$$

If we have reason to believe an $n \times n$ system has a unique solution, is there a formula for that solution? Here is a what happens when $n = 1$:

$$ax_1 = b, \quad x_1 = \frac{b}{a}$$

We know it has a unique solution if $a \neq 0$.

If we have reason to believe an $n \times n$ system has a unique solution, is there a formula for that solution? Here is a what happens when $n = 1$:

$$ax_1 = b, \quad x_1 = \frac{b}{a}$$

We know it has a unique solution if $a \neq 0$. Now consider

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

If we have reason to believe an $n \times n$ system has a unique solution, is there a formula for that solution? Here is a what happens when $n = 1$:

$$ax_1 = b, \quad x_1 = \frac{b}{a}$$

We know it has a unique solution if $a \neq 0$. Now consider

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

If we multiply the first equation by a_{21} and the second equation by a_{11} and subtract the first from the second we get

$$(a_{11}a_{22} - a_{21}a_{12})x_2 = b_2a_{11} - b_1a_{21} \text{ and so, } x_2 = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{21}a_{12}}$$

If we have reason to believe an $n \times n$ system has a unique solution, is there a formula for that solution? Here is a what happens when $n = 1$:

$$ax_1 = b, \quad x_1 = \frac{b}{a}$$

We know it has a unique solution if $a \neq 0$. Now consider

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

If we multiply the first equation by a_{21} and the second equation by a_{11} and subtract the first from the second we get

$$(a_{11}a_{22} - a_{21}a_{12})x_2 = b_2a_{11} - b_1a_{21} \text{ and so, } x_2 = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{21}a_{12}}$$

We can do a similar trick to eliminate x_2 from the first equation and get $x_1 = \dfrac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}}$.

If we have reason to believe an $n \times n$ system has a unique solution, is there a formula for that solution? Here is a what happens when $n = 1$:

$$ax_1 = b, \quad x_1 = \frac{b}{a}$$

We know it has a unique solution if $a \neq 0$. Now consider

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

If we multiply the first equation by a_{21} and the second equation by a_{11} and subtract the first from the second we get

$$(a_{11}a_{22} - a_{21}a_{12})x_2 = b_2a_{11} - b_1a_{21} \text{ and so, } x_2 = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{21}a_{12}}$$

We can do a similar trick to eliminate x_2 from the first equation and get $x_1 = \dfrac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}}$. This works if $a_{11}a_{22} - a_{21}a_{12} \neq 0$.

The denominator $a_{11}a_{22} - a_{21}a_{12}$ is called the *determinant* of the 2×2 system. Historically, this was because it *determines* whether the system had a unique solution.

The denominator $a_{11}a_{22} - a_{21}a_{12}$ is called the *determinant* of the 2×2 system. Historically, this was because it *determines* whether the system had a unique solution.

This same sort of thing can be done for larger $n \times n$ systems, but the formulas quickly get way out of hand.

The denominator $a_{11}a_{22} - a_{21}a_{12}$ is called the *determinant* of the 2×2 system. Historically, this was because it *determines* whether the system had a unique solution.

This same sort of thing can be done for larger $n \times n$ systems, but the formulas quickly get way out of hand. For 3×3 systems, the variables can again be solved for as fractions with this denominator:

$$D = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

The denominator $a_{11}a_{22} - a_{21}a_{12}$ is called the *determinant* of the 2×2 system. Historically, this was because it *determines* whether the system had a unique solution.

This same sort of thing can be done for larger $n \times n$ systems, but the formulas quickly get way out of hand. For 3×3 systems, the variables can again be solved for as fractions with this denominator:

$$D = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

The numerators are similarly complex.

The denominator $a_{11}a_{22} - a_{21}a_{12}$ is called the *determinant* of the 2×2 system. Historically, this was because it *determines* whether the system had a unique solution.

This same sort of thing can be done for larger $n \times n$ systems, but the formulas quickly get way out of hand. For 3×3 systems, the variables can again be solved for as fractions with this denominator:

$$D = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

The numerators are similarly complex. For 4×4 systems there are 24 terms with four factors in each term.

The denominator $a_{11}a_{22} - a_{21}a_{12}$ is called the *determinant* of the 2×2 system. Historically, this was because it *determines* whether the system had a unique solution.

This same sort of thing can be done for larger $n \times n$ systems, but the formulas quickly get way out of hand. For 3×3 systems, the variables can again be solved for as fractions with this denominator:

$$D = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

The numerators are similarly complex. For 4×4 systems there are 24 terms with four factors in each term.

In the formula for D above, each term is a product $a_{1i}a_{2j}a_{3k}$ where i, j, k is a permutation of $1, 2, 3$.

The denominator $a_{11}a_{22} - a_{21}a_{12}$ is called the *determinant* of the 2×2 system. Historically, this was because it *determines* whether the system had a unique solution.

This same sort of thing can be done for larger $n \times n$ systems, but the formulas quickly get way out of hand. For 3×3 systems, the variables can again be solved for as fractions with this denominator:

$$D = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

The numerators are similarly complex. For 4×4 systems there are 24 terms with four factors in each term.

In the formula for D above, each term is a product $a_{1i}a_{2j}a_{3k}$ where i, j, k is a permutation of $1, 2, 3$. If one permutation comes from another by a single exchange of 2 positions, then the sign of the product is changed.

Thus, we define the determinants of an $n \times n$ matrix in a more computable way.

Thus, we define the determinants of an $n \times n$ matrix in a more computable way.

First, some notation and terminology. If A is a square matrix we write $\det(A)$ for its determinant.

Thus, we define the determinants of an $n \times n$ matrix in a more computable way.

First, some notation and terminology. If A is a square matrix we write $\det(A)$ for its determinant.

$$\text{If} \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Thus, we define the determinants of an $n \times n$ matrix in a more computable way.

First, some notation and terminology. If A is a square matrix we write $\det(A)$ for its determinant.

$$\text{If } \; A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

then we write

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad \text{for its determinant.}$$

For a 1×1 matrix define: if $A = \begin{pmatrix} a_{11} \end{pmatrix}$ then $\det(A) = a_{11}$. For a 2×2 matrix define:

$$\text{if} \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{then} \quad \det(A) = a_{11}a_{22} - a_{21}a_{12}.$$

Minors

If A is an $n \times n$ matrix, the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A will be denoted M_{ij}. This is called the *ij-minor* of A. For example, if

$$A = \begin{pmatrix} 2 & 2 & 4 \\ 0 & 5 & -3 \\ 2 & -1 & 4 \end{pmatrix} \quad \text{then} \quad M_{23} = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$$

Cofactors

If A is an $n \times n$ matrix and M_{ij} is the ij-minor, then $A_{ij} = (-1)^{i+j} \det(M_{ij})$ is called the *ij-cofactor* of A.

Cofactors

If A is an $n \times n$ matrix and M_{ij} is the ij-minor, then $A_{ij} = (-1)^{i+j} \det(M_{ij})$ is called the *ij-cofactor* of A. The example from the previous page: if

$$A = \begin{pmatrix} 2 & 2 & 4 \\ 0 & 5 & -3 \\ 2 & -1 & 4 \end{pmatrix} \quad \text{then} \quad A_{23} = -\begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix} = 6$$

Cofactors

If A is an $n \times n$ matrix and M_{ij} is the ij-minor, then $A_{ij} = (-1)^{i+j} \det(M_{ij})$ is called the *ij-cofactor* of A. The example from the previous page: if

$$A = \begin{pmatrix} 2 & 2 & 4 \\ 0 & 5 & -3 \\ 2 & -1 & 4 \end{pmatrix} \text{ then } A_{23} = -\begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix} = 6$$

Now we can describe a computation that produces the determinant of a matrix A.

Definition

The determinant of an $n \times n$ matrix A is a scalar associated to A that is computed recursively by

$$\det(A) = \begin{cases} a_{11} & \text{if } n = 1, \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} & \text{if } n > 1. \end{cases}$$

For example, if $A = \begin{pmatrix} 2 & 2 & 4 \\ 0 & 5 & -3 \\ 2 & -1 & 4 \end{pmatrix}$ then the determinant of A is

$$2\begin{vmatrix} 5 & -3 \\ -1 & 4 \end{vmatrix} + 2\left(-\begin{vmatrix} 0 & -3 \\ 2 & 4 \end{vmatrix}\right) + 4\begin{vmatrix} 0 & 5 \\ 2 & -1 \end{vmatrix}$$

$$= 2(17) + 2(-6) + 4(-10) = -18$$

For example, if $A = \begin{pmatrix} 2 & 2 & 4 \\ 0 & 5 & -3 \\ 2 & -1 & 4 \end{pmatrix}$ then the determinant of A is

$$2\begin{vmatrix} 5 & -3 \\ -1 & 4 \end{vmatrix} + 2\left(-\begin{vmatrix} 0 & -3 \\ 2 & 4 \end{vmatrix}\right) + 4\begin{vmatrix} 0 & 5 \\ 2 & -1 \end{vmatrix}$$
$$= 2(17) + 2(-6) + 4(-10) = -18$$

Some useful properties of determinants

For example, if $A = \begin{pmatrix} 2 & 2 & 4 \\ 0 & 5 & -3 \\ 2 & -1 & 4 \end{pmatrix}$ then the determinant of A is

$$2\begin{vmatrix} 5 & -3 \\ -1 & 4 \end{vmatrix} + 2\left(-\begin{vmatrix} 0 & -3 \\ 2 & 4 \end{vmatrix}\right) + 4\begin{vmatrix} 0 & 5 \\ 2 & -1 \end{vmatrix}$$
$$= 2(17) + 2(-6) + 4(-10) = -18$$

Some useful properties of determinants

Theorem

The calculation of $\det(A)$ *can be obtained by using any row of* A, *that is,*

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}.$$

For example, if $A = \begin{pmatrix} 2 & 2 & 4 \\ 0 & 5 & -3 \\ 2 & -1 & 4 \end{pmatrix}$ then the determinant of A is

$$0\left(-\begin{vmatrix} 2 & 4 \\ -1 & 4 \end{vmatrix}\right) + 5\begin{vmatrix} 2 & 4 \\ 2 & 4 \end{vmatrix} + (-3)\left(-\begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix}\right)$$

$$= 0 + 5(0) - 3(6) = -18$$

For example, if $A = \begin{pmatrix} 2 & 2 & 4 \\ 0 & 5 & -3 \\ 2 & -1 & 4 \end{pmatrix}$ then the determinant of A is

$$0\left(-\begin{vmatrix} 2 & 4 \\ -1 & 4 \end{vmatrix}\right) + 5\begin{vmatrix} 2 & 4 \\ 2 & 4 \end{vmatrix} + (-3)\left(-\begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix}\right)$$
$$= 0 + 5(0) - 3(6) = -18$$

The following pattern might be useful: $\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$

For example, if $A = \begin{pmatrix} 2 & 2 & 4 \\ 0 & 5 & -3 \\ 2 & -1 & 4 \end{pmatrix}$ then the determinant of A is

$$0\left(-\begin{vmatrix} 2 & 4 \\ -1 & 4 \end{vmatrix}\right) + 5\begin{vmatrix} 2 & 4 \\ 2 & 4 \end{vmatrix} + (-3)\left(-\begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix}\right)$$

$$= 0 + 5(0) - 3(6) = -18$$

The following pattern might be useful: $\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$

A matrix A with a row that is all zeros has $\det(A) = 0$. This is clear since we can use that row to compute the determinant.

A matrix A is *upper triangular* if every entry below the diagonal from a_{11} to a_{nn} is zero. That is, $a_{ij} = 0$ for every entry with $i > j$.

A matrix A is *upper triangular* if every entry below the diagonal from a_{11} to a_{nn} is zero. That is, $a_{ij} = 0$ for every entry with $i > j$. A matrix is *lower triangular* if $a_{ij} = 0$ for every entry with $i < j$.

A matrix A is *upper triangular* if every entry below the diagonal from a_{11} to a_{nn} is zero. That is, $a_{ij} = 0$ for every entry with $i > j$. A matrix is *lower triangular* if $a_{ij} = 0$ for every entry with $i < j$.

If A is triangular (either upper or lower) then its determinant is the product of the diagonal entries: $a_{11}a_{22} \cdots a_{nn}$.

A matrix A is *upper triangular* if every entry below the diagonal from a_{11} to a_{nn} is zero. That is, $a_{ij} = 0$ for every entry with $i > j$. A matrix is *lower triangular* if $a_{ij} = 0$ for every entry with $i < j$.

If A is triangular (either upper or lower) then its determinant is the product of the diagonal entries: $a_{11}a_{22} \cdots a_{nn}$.

The effect of elementary row operations

A matrix A is *upper triangular* if every entry below the diagonal from a_{11} to a_{nn} is zero. That is, $a_{ij} = 0$ for every entry with $i > j$. A matrix is *lower triangular* if $a_{ij} = 0$ for every entry with $i < j$.

If A is triangular (either upper or lower) then its determinant is the product of the diagonal entries: $a_{11}a_{22}\cdots a_{nn}$.

The effect of elementary row operations

One way to calculate $\det(A)$ is to reduce it to a triangular matrix using EROs.

A matrix A is *upper triangular* if every entry below the diagonal from a_{11} to a_{nn} is zero. That is, $a_{ij} = 0$ for every entry with $i > j$. A matrix is *lower triangular* if $a_{ij} = 0$ for every entry with $i < j$.

If A is triangular (either upper or lower) then its determinant is the product of the diagonal entries: $a_{11}a_{22} \cdots a_{nn}$.

The effect of elementary row operations

One way to calculate $\det(A)$ is to reduce it to a triangular matrix using EROs. This works as long as we keep track of how these EROs change the determinant.

A matrix A is *upper triangular* if every entry below the diagonal from a_{11} to a_{nn} is zero. That is, $a_{ij} = 0$ for every entry with $i > j$. A matrix is *lower triangular* if $a_{ij} = 0$ for every entry with $i < j$.

If A is triangular (either upper or lower) then its determinant is the product of the diagonal entries: $a_{11}a_{22} \cdots a_{nn}$.

The effect of elementary row operations

One way to calculate $\det(A)$ is to reduce it to a triangular matrix using EROs. This works as long as we keep track of how these EROs change the determinant.

1. Type I: If two rows of A are exchanged, then the determinant changes sign.

A matrix A is *upper triangular* if every entry below the diagonal from a_{11} to a_{nn} is zero. That is, $a_{ij} = 0$ for every entry with $i > j$. A matrix is *lower triangular* if $a_{ij} = 0$ for every entry with $i < j$.

If A is triangular (either upper or lower) then its determinant is the product of the diagonal entries: $a_{11}a_{22}\cdots a_{nn}$.

The effect of elementary row operations

One way to calculate $\det(A)$ is to reduce it to a triangular matrix using EROs. This works as long as we keep track of how these EROs change the determinant.

1. Type I: If two rows of A are exchanged, then the determinant changes sign.
2. Type II: If a row of A is multiplied by α then the determinant of the new matrix is $\alpha \det(A)$.

A matrix A is *upper triangular* if every entry below the diagonal from a_{11} to a_{nn} is zero. That is, $a_{ij} = 0$ for every entry with $i > j$. A matrix is *lower triangular* if $a_{ij} = 0$ for every entry with $i < j$.

If A is triangular (either upper or lower) then its determinant is the product of the diagonal entries: $a_{11}a_{22}\cdots a_{nn}$.

The effect of elementary row operations

One way to calculate $\det(A)$ is to reduce it to a triangular matrix using EROs. This works as long as we keep track of how these EROs change the determinant.

1. Type I: If two rows of A are exchanged, then the determinant changes sign.
2. Type II: If a row of A is multiplied by α then the determinant of the new matrix is $\alpha \det(A)$.
3. Type III: If a row is changed by adding to it a multiple of another row, the determinant is not changed.

Using EROs to calculate $\det(A)$:

$$\begin{vmatrix} 2 & 2 & 4 \\ 0 & 5 & -3 \\ 2 & -1 & 4 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 2 \\ 0 & 5 & -3 \\ 2 & -1 & 4 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 2 \\ 0 & 5 & -3 \\ 0 & -3 & 0 \end{vmatrix}$$

Using EROs to calculate $\det(A)$:

$$\begin{vmatrix} 2 & 2 & 4 \\ 0 & 5 & -3 \\ 2 & -1 & 4 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 2 \\ 0 & 5 & -3 \\ 2 & -1 & 4 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 2 \\ 0 & 5 & -3 \\ 0 & -3 & 0 \end{vmatrix}$$

Continuing

$$= -2 \begin{vmatrix} 1 & 1 & 2 \\ 0 & -3 & 0 \\ 0 & 5 & -3 \end{vmatrix} = -2(-3) \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 5 & -3 \end{vmatrix} = 6 \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{vmatrix} = -18$$

Using EROs to calculate $\det(A)$:

$$\begin{vmatrix} 2 & 2 & 4 \\ 0 & 5 & -3 \\ 2 & -1 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 2 \\ 0 & 5 & -3 \\ 2 & -1 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 2 \\ 0 & 5 & -3 \\ 0 & -3 & 0 \end{vmatrix}$$

Continuing

$$= -2\begin{vmatrix} 1 & 1 & 2 \\ 0 & -3 & 0 \\ 0 & 5 & -3 \end{vmatrix} = -2(-3)\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 5 & -3 \end{vmatrix} = 6\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{vmatrix} = -18$$

A side note: any function that assigns a scalar to a matrix and satisfies those three ERO conditions, plus the additional condition that it assigns the value 1 to the identity matrix, must in fact be the determinant.

Using EROs to calculate $\det(A)$:

$$\begin{vmatrix} 2 & 2 & 4 \\ 0 & 5 & -3 \\ 2 & -1 & 4 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 2 \\ 0 & 5 & -3 \\ 2 & -1 & 4 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 2 \\ 0 & 5 & -3 \\ 0 & -3 & 0 \end{vmatrix}$$

Continuing

$$= -2 \begin{vmatrix} 1 & 1 & 2 \\ 0 & -3 & 0 \\ 0 & 5 & -3 \end{vmatrix} = -2(-3) \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 5 & -3 \end{vmatrix} = 6 \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{vmatrix} = -18$$

A side note: any function that assigns a scalar to a matrix and satisfies those three ERO conditions, plus the additional condition that it assigns the value 1 to the identity matrix, must in fact be the determinant.

Because of this it can be proved that we can calculate the determinant by using cofactors along a column instead of a row:

Using EROs to calculate $\det(A)$:

$$\begin{vmatrix} 2 & 2 & 4 \\ 0 & 5 & -3 \\ 2 & -1 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 2 \\ 0 & 5 & -3 \\ 2 & -1 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 2 \\ 0 & 5 & -3 \\ 0 & -3 & 0 \end{vmatrix}$$

Continuing

$$= -2\begin{vmatrix} 1 & 1 & 2 \\ 0 & -3 & 0 \\ 0 & 5 & -3 \end{vmatrix} = -2(-3)\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 5 & -3 \end{vmatrix} = 6\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{vmatrix} = -18$$

A side note: any function that assigns a scalar to a matrix and satisfies those three ERO conditions, plus the additional condition that it assigns the value 1 to the identity matrix, must in fact be the determinant.

Because of this it can be proved that we can calculate the determinant by using cofactors along a column instead of a row:

$$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}.$$

From this, it can be proved that the transpose of A has the same determinant as A: $\det(A^T) = \det(A)$.

From this, it can be proved that the transpose of A has the same determinant as A: $\det(A^T) = \det(A)$.

If a matrix has one row that is a multiple of another row then its determinant is zero:

From this, it can be proved that the transpose of A has the same determinant as A: $\det(A^T) = \det(A)$.

If a matrix has one row that is a multiple of another row then its determinant is zero: one type III ERO will turn it into a matrix with a row of zeros.

From this, it can be proved that the transpose of A has the same determinant as A: $\det(A^T) = \det(A)$.

If a matrix has one row that is a multiple of another row then its determinant is zero: one type III ERO will turn it into a matrix with a row of zeros.

Because we can calculate determinants using cofactors along a column, it follows that if any column is all 0's then the determinant is 0.

From this, it can be proved that the transpose of A has the same determinant as A: $\det(A^T) = \det(A)$.

If a matrix has one row that is a multiple of another row then its determinant is zero: one type III ERO will turn it into a matrix with a row of zeros.

Because we can calculate determinants using cofactors along a column, it follows that if any column is all 0's then the determinant is 0. It also follows we get the same behavior under *elementary column operations* as we have under EROs.

From this, it can be proved that the transpose of A has the same determinant as A: $\det(A^T) = \det(A)$.

If a matrix has one row that is a multiple of another row then its determinant is zero: one type III ERO will turn it into a matrix with a row of zeros.

Because we can calculate determinants using cofactors along a column, it follows that if any column is all 0's then the determinant is 0. It also follows we get the same behavior under *elementary column operations* as we have under EROs. And if one column is a scalar multiple of another column, then the determinant is 0.

From this, it can be proved that the transpose of A has the same determinant as A: $\det(A^T) = \det(A)$.

If a matrix has one row that is a multiple of another row then its determinant is zero: one type III ERO will turn it into a matrix with a row of zeros.

Because we can calculate determinants using cofactors along a column, it follows that if any column is all 0's then the determinant is 0. It also follows we get the same behavior under *elementary column operations* as we have under EROs. And if one column is a scalar multiple of another column, then the determinant is 0.

If a matrix has the form $A = \begin{pmatrix} A_{11} & A_{12} \\ \mathcal{O} & A_{22} \end{pmatrix}$ or the form $A = \begin{pmatrix} A_{11} & \mathcal{O} \\ A_{21} & A_{22} \end{pmatrix}$ then $\det(A) = \det(A_{11})\det(A_{22})$. Of course, this requires A_{11} and A_{22} to be square matrices.