Partitioned Matrices

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   Example: $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & -1 \end{pmatrix} \left(\begin{array}{cc|r} 2 & 5 & 1 \\ 1 & 0 & -1 \end{array} \right) = \left(\begin{array}{rr|r} 4 & 5 & -1 \\ 10 & 15 & -1 \\ -1 & 0 & 1 \end{array} \right)$$

2. $$\left(\frac{A_1}{A_2} \right) B = \left(\frac{A_1 B}{A_2 B} \right)$$

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3. $$\left(\begin{array}{c|c} A_1 & A_2 \end{array}\right) \left(\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right) = A_1 B_1 + A_2 B_2$$

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Example:

$$\left(\begin{array}{c|c} 1 & 2 \\ 3 & 4 \end{array}\right) \left(\begin{array}{cc} 2 & 3 \\ \hline -1 & 3 \end{array}\right) = \left(\begin{array}{cc} 2 & 3 \\ 6 & 9 \end{array}\right) + \left(\begin{array}{cc} -2 & 6 \\ -4 & 12 \end{array}\right) = \left(\begin{array}{cc} 0 & 9 \\ 2 & 21 \end{array}\right)$$

Finally,

4. Either

$$\left(\begin{array}{c|c} \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 \end{array} \right) \left(\begin{array}{c|c} B_{11} & \mathcal{O} \\ \hline B_{21} & B_{22} \end{array} \right) = \left(\begin{array}{c|c} \vec{\mathbf{a}}_1 B_{11} + \vec{\mathbf{a}}_2 B_{21} & \vec{\mathbf{a}}_2 B_{22} \end{array} \right)$$

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Example: $\left(\begin{array}{rr|r} 1 & 2 & 3 \\ -1 & 3 & 4 \\ \hline 0 & 0 & 5 \end{array} \right) \left(\begin{array}{c} 2 \\ 3 \\ \hline 1 \end{array} \right) = \left(\begin{array}{c} 8+3 \\ 7+4 \\ \hline 5 \end{array} \right)$

Review

System of linear equations (aka "system"):

Example: $$\begin{cases} \phantom{3x_1 + {}} 3x_2 - 3x_3 + 3x_4 = 0 \\ x_1 + 2x_2 \phantom{{} - 3x_3} + x_4 = 1 \\ 3x_1 + 6x_2 \phantom{{} - 3x_3} + 5x_4 = -3 \end{cases}$$

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Augmented matrix:

$$\text{Example:}\quad \left(\begin{array}{cccc|c} 0 & 3 & -3 & 3 & 0 \\ 1 & 2 & 0 & 1 & 1 \\ 3 & 6 & 0 & 5 & -3 \end{array}\right)$$

Gauss-Jordan reduction:

$$\left(\begin{array}{cccc|c} 0 & 3 & -3 & 3 & 0 \\ 1 & 2 & 0 & 1 & 1 \\ 3 & 6 & 0 & 5 & -3 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 3 & -3 & 3 & 0 \\ 3 & 6 & 0 & 5 & -3 \end{array}\right)$$

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$$\xrightarrow{R_3 - 3R_1} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 3 & -3 & 3 & 0 \\ 0 & 0 & 0 & 2 & -6 \end{array}\right) \xrightarrow[(1/2)R_3]{(1/3)R_2} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right)$$

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We now know there is at least one solution: because there is no row with all zeros left of the bar and nonzero to the right. The leading variables are x_1, x_2 and x_4. Since we have a free variable x_3, there are infinitely many solutions.

Echelon form and reduced echelon form:

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right) \xrightarrow[R_1-R_3]{R_2-R_3} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 4 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right)$$

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$$\xrightarrow{R_1-2R_2} \left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right) \xrightarrow{\text{yields}} \begin{cases} x_1 \quad + 2x_3 = -2 \\ \quad x_2 - \ x_3 = 3 \\ \qquad\qquad x_4 = -3 \end{cases}$$

Solving:

$$\left.\begin{array}{l} x_1 = -2 - 2x_3 \\ x_2 = 3 + x_3 \\ x_4 = -3 \end{array}\right\} \xrightarrow{\text{and so,}} \begin{cases} x_3 = \alpha \\ x_1 = -2 - 2\alpha \\ x_2 = 3 + \alpha \\ x_4 = -3 \end{cases}$$

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Whence the solutions are

$$(-2 - 2\alpha, 3 + \alpha, \alpha, -3) \quad \text{or} \quad \begin{pmatrix} -2 - 2\alpha \\ 3 + \alpha \\ \alpha \\ -3 \end{pmatrix}$$

Arithmetic with matrices.

Scalar multiplication: if $A = (a_{ij})$ then $\alpha A = (\alpha a_{ij})$.

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Addition: If $A = (a_{ij})_{n\times k}$ and $B = (b_{ij})_{n\times k}$ then $A + B = (a_{ij} + b_{ij})_{n\times k}$.

$$\text{Example: } \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 5 & -2 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 1 & 0 \end{pmatrix}$$

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Alternatively:

$$\text{If } A = \begin{pmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_n \end{pmatrix} \text{ and } B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{pmatrix}$$

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$$\text{Example: } \begin{pmatrix} 1 & 3 & -2 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 6 \\ 0 & -2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & -6 \\ -1 & -1 \end{pmatrix}$$

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Any invertible matrix A is a product of elementary matrices and A^{-1} is the product of their inverses in the opposite order.

We can find the inverse of A (or perhaps discover it has none) by reducing the matrix $(A \mid I)$ to reduced echelon form.

Example: A is the matrix on the left below.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 5 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2-2R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

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$$\xrightarrow{R_1-R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1-2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -2 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

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so

$$A^{-1} = \left(\begin{array}{ccc} 5 & -2 & -1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$