More on Matrices

D. H. Luecking

02 Feb 2024

Powers of matrices

Multipling two matrices is possible only if the width of the first matches the height of the second. So multiplying AA requires A to be a square matrix.

Powers of matrices

Multipling two matrices is possible only if the width of the first matches the height of the second. So multiplying AA requires A to be a square matrix. The result of AA is the same height as the first matrix and the same width as the second, so AA must be the same size as A and we can continue multipying: AAA, $AAAA$ and so on.

Powers of matrices

Multipling two matrices is possible only if the width of the first matches the height of the second. So multiplying AA requires A to be a square matrix. The result of AA is the same height as the first matrix and the same width as the second, so AA must be the same size as A and we can continue multipying: AAA, $AAAA$ and so on.

We abbreviate $AA = A^2$ and $AAA = A^3$, etc. So $A^n = AA \cdots A$ for a total of n factors.

Powers of matrices

Multipling two matrices is possible only if the width of the first matches the height of the second. So multiplying AA requires A to be a square matrix. The result of AA is the same height as the first matrix and the same width as the second, so AA must be the same size as A and we can continue multipying: AAA, $AAAA$ and so on.

We abbreviate $AA = A^2$ and $AAA = A^3$, etc. So $A^n = AA \cdots A$ for a total of n factors. If $k + m = n$ then $A^k A^m = A^n$. This formula extends to include negative integers if we make the following definitiions:

Powers of matrices

Multipling two matrices is possible only if the width of the first matches the height of the second. So multiplying AA requires A to be a square matrix. The result of AA is the same height as the first matrix and the same width as the second, so AA must be the same size as A and we can continue multipying: AAA, $AAAA$ and so on.

We abbreviate $AA = A^2$ and $AAA = A^3$, etc. So $A^n = AA \cdots A$ for a total of n factors. If $k + m = n$ then $A^k A^m = A^n$. This formula extends to include negative integers if we make the following definitiions:

Definition

For a square matrix A we let $A^0 = I$ where I is the identity matrix (same size as A). If A is invertible and $n > 0$, $A^{-n} = (A^{-1})^n$.

Powers of matrices

Multipling two matrices is possible only if the width of the first matches the height of the second. So multiplying AA requires A to be a square matrix. The result of AA is the same height as the first matrix and the same width as the second, so AA must be the same size as A and we can continue multipying: AAA, $AAAA$ and so on.

We abbreviate $AA = A^2$ and $AAA = A^3$, etc. So $A^n = AA \cdots A$ for a total of n factors. If $k + m = n$ then $A^k A^m = A^n$. This formula extends to include negative integers if we make the following definitiions:

Definition

For a square matrix A we let $A^0 = I$ where I is the identity matrix (same size as A). If A is invertible and $n > 0$, $A^{-n} = (A^{-1})^n$.

We have the following properties for any invertible matrix A:

1. For any integers k and m (positive negative or zero) $A^k A^m = A^{k+m}$.
2. For any integers k and m (positive negative or zero) $(A^k)^m = A^{km}$. In particular, if $m = -1$ we get $(A^k)^{-1} = A^{-k}$.

This example from the book illustrates the use of powers of a matrix. In a certain town with 10000 women the following trend has been observed: Every year 30% of the married women get a divorce and 20% of the single women get married.

This example from the book illustrates the use of powers of a matrix. In a certain town with 10000 women the following trend has been observed: Every year 30% of the married women get a divorce and 20% of the single women get married. We can express this with the following set of equations, in which x_0 is the number of married women and y_0 is the number of single women this year, while x_1 and y_1 are the numbers for next year. Then

$$0.7x_0 + 0.2y_0 = x_1$$
$$0.3x_0 + 0.8y_0 = y_1$$

This example from the book illustrates the use of powers of a matrix. In a certain town with 10000 women the following trend has been observed: Every year 30% of the married women get a divorce and 20% of the single women get married. We can express this with the following set of equations, in which x_0 is the number of married women and y_0 is the number of single women this year, while x_1 and y_1 are the numbers for next year. Then

$$0.7x_0 + 0.2y_0 = x_1$$
$$0.3x_0 + 0.8y_0 = y_1$$

Written as a matrix equation, this is

$$\begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

This example from the book illustrates the use of powers of a matrix. In a certain town with 10000 women the following trend has been observed: Every year 30% of the married women get a divorce and 20% of the single women get married. We can express this with the following set of equations, in which x_0 is the number of married women and y_0 is the number of single women this year, while x_1 and y_1 are the numbers for next year. Then

$$0.7x_0 + 0.2y_0 = x_1$$
$$0.3x_0 + 0.8y_0 = y_1$$

Written as a matrix equation, this is

$$\begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

This allows us to predict the number of married and unmarried women for any year in the future.

This example from the book illustrates the use of powers of a matrix. In a certain town with 10000 women the following trend has been observed: Every year 30% of the married women get a divorce and 20% of the single women get married. We can express this with the following set of equations, in which x_0 is the number of married women and y_0 is the number of single women this year, while x_1 and y_1 are the numbers for next year. Then

$$0.7x_0 + 0.2y_0 = x_1$$
$$0.3x_0 + 0.8y_0 = y_1$$

Written as a matrix equation, this is

$$\begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

This allows us to predict the number of married and unmarried women for any year in the future.

Let $A = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}$ and suppose $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 8000 \\ 2000 \end{pmatrix}$

Then the numbers next year will be

$$A \left(\begin{array}{c} 8000 \\ 2000 \end{array} \right) = \left(\begin{array}{c} 6000 \\ 4000 \end{array} \right)$$

Then the numbers next year will be

$$A \begin{pmatrix} 8000 \\ 2000 \end{pmatrix} = \begin{pmatrix} 6000 \\ 4000 \end{pmatrix}$$

And then after 2 and 3 years the numbers will be

$$A^2 \begin{pmatrix} 8000 \\ 2000 \end{pmatrix} = \begin{pmatrix} 5000 \\ 5000 \end{pmatrix} \quad \text{and} \quad A^3 \begin{pmatrix} 8000 \\ 2000 \end{pmatrix} = \begin{pmatrix} 4500 \\ 5500 \end{pmatrix}$$

Then the numbers next year will be

$$A \begin{pmatrix} 8000 \\ 2000 \end{pmatrix} = \begin{pmatrix} 6000 \\ 4000 \end{pmatrix}$$

And then after 2 and 3 years the numbers will be

$$A^2 \begin{pmatrix} 8000 \\ 2000 \end{pmatrix} = \begin{pmatrix} 5000 \\ 5000 \end{pmatrix} \quad \text{and} \quad A^3 \begin{pmatrix} 8000 \\ 2000 \end{pmatrix} = \begin{pmatrix} 4500 \\ 5500 \end{pmatrix}$$

We will see much later that

$$A^n \begin{pmatrix} 8000 \\ 2000 \end{pmatrix} \quad \text{approaches} \quad \begin{pmatrix} 4000 \\ 6000 \end{pmatrix}$$

Partitioned matrices

If we take any matrix A and draw horizontal lines between some rows and vertical lines between some columns, this will divide A into submatrices

Partitioned matrices

If we take any matrix A and draw horizontal lines between some rows and vertical lines between some columns, this will divide A into submatrices and sometimes it is useful think of A in terms of these partial matrices.

Partitioned matrices

If we take any matrix A and draw horizontal lines between some rows and vertical lines between some columns, this will divide A into submatrices and sometimes it is useful think of A in terms of these partial matrices.

For example,

$$\left(\begin{array}{cc|cc} 1 & -2 & 1 & 3 \\ 2 & 1 & 0 & 4 \\ \hline 4 & 6 & 1 & -3 \end{array}\right) = \left(\begin{array}{cc} C_{11} & C_{12} \\ C_{21} & C_{22} \end{array}\right)$$

Where C_{ij} are matrices.

Partitioned matrices

If we take any matrix A and draw horizontal lines between some rows and vertical lines between some columns, this will divide A into submatrices and sometimes it is useful think of A in terms of these partial matrices.

For example,

$$\left(\begin{array}{cc|cc} 1 & -2 & 1 & 3 \\ 2 & 1 & 0 & 4 \\ \hline 4 & 6 & 1 & -3 \end{array}\right) = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

Where C_{ij} are matrices. Note that matrices in a given row must be the same height and matrices in a given column must be the same width.

We've already seen something like this where we think of a matrix as made up of its columns:

$$B = \left(\begin{array}{r|r|r} -1 & 2 & 4 \\ 3 & 1 & 0 \end{array} \right) = \left(\begin{array}{ccc} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{array} \right)$$

We've already seen something like this where we think of a matrix as made up of its columns:

$$B = \left(\begin{array}{r|r|r} -1 & 2 & 4 \\ 3 & 1 & 0 \end{array} \right) = \left(\begin{array}{ccc} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{array} \right)$$

As part of the definition of matrix multiplication we saw that

$$A \left(\begin{array}{ccc} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{array} \right) = \left(\begin{array}{ccc} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{array} \right)$$

This is true even if the parts contain more that one column: if B_j are matrices with all the same height then

$$A \left(\begin{array}{cccc} B_1 & B_2 & \cdots & B_m \end{array} \right) = \left(\begin{array}{cccc} AB_1 & AB_2 & \cdots & AB_m \end{array} \right)$$

We've already seen something like this where we think of a matrix as made up of its columns:

$$B = \left(\begin{array}{r|r|r} -1 & 2 & 4 \\ 3 & 1 & 0 \end{array} \right) = \left(\begin{array}{ccc} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{array} \right)$$

As part of the definition of matrix multiplication we saw that

$$A \left(\begin{array}{ccc} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{array} \right) = \left(\begin{array}{ccc} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{array} \right)$$

This is true even if the parts contain more that one column: if B_j are matrices with all the same height then

$$A \left(\begin{array}{cccc} B_1 & B_2 & \cdots & B_m \end{array} \right) = \left(\begin{array}{cccc} AB_1 & AB_2 & \cdots & AB_m \end{array} \right)$$

We have already mentioned this when we talked about finding inverses:

$$M \left(\begin{array}{c|c} A & I \end{array} \right) = \left(\begin{array}{c|c} MA & MI \end{array} \right)$$

There is a similar process that works with partitions into rows:

$$\begin{pmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_n \end{pmatrix} B = \begin{pmatrix} \vec{\mathbf{a}}_1 B \\ \vec{\mathbf{a}}_2 B \\ \vdots \\ \vec{\mathbf{a}}_n B \end{pmatrix}$$

There is a similar process that works with partitions into rows:

$$\begin{pmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_n \end{pmatrix} B = \begin{pmatrix} \vec{\mathbf{a}}_1 B \\ \vec{\mathbf{a}}_2 B \\ \vdots \\ \vec{\mathbf{a}}_n B \end{pmatrix}$$

And we also don't need the parts to be rows: if A_j are matrices with same width then

$$\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} B = \begin{pmatrix} A_1 B \\ A_2 B \\ \vdots \\ A_n B \end{pmatrix}$$

We also have formulas in which both A and B can be partitioned.

We also have formulas in which both A and B can be partitioned. The most common example is where A is partitioned into its columns and B into rows:

$$AB = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_k \end{pmatrix} \begin{pmatrix} \vec{\mathbf{b}}_1 \\ \vec{\mathbf{b}}_2 \\ \vdots \\ \vec{\mathbf{b}}_k \end{pmatrix} = \sum_{j=1}^{k} \mathbf{a}_j \vec{\mathbf{b}}_j$$

We also have formulas in which both A and B can be partitioned. The most common example is where A is partitioned into its columns and B into rows:

$$AB = \left(\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_k \end{array} \right) \left(\begin{array}{c} \vec{\mathbf{b}}_1 \\ \vec{\mathbf{b}}_2 \\ \vdots \\ \vec{\mathbf{b}}_k \end{array} \right) = \sum_{j=1}^{k} \mathbf{a}_j \vec{\mathbf{b}}_j$$

This is called the *outer product expansion* of AB. All the terms in the sum are outer products.

Again, we don't need rows and columns.

Again, we don't need rows and columns. If A_j and B_j are matrices where all A_j have the same height, all B_j have the same width and the width of each A_j is the same as the height of the corresponding B_j,

then

$$\begin{pmatrix} A_1 & A_2 & \cdots & A_k \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_k \end{pmatrix} = \sum_{j=1}^{k} A_j B_j$$

Finally, to give an inkling how far we can take this

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

provided the sizes are such that all the products can be performed and all the rows and columns line up.