# Matrix Inverses

D. H. Luecking

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## Solving multiple systems

Suppose we have two systems with same left sides but different right sides, such as

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$$\left(\begin{array}{cc|cc} 1 & 2 & 3 & 1 \\ 2 & 5 & 4 & 0 \end{array}\right) \xrightarrow{R_2-2R_1} \left(\begin{array}{cc|rr} 1 & 2 & 3 & 1 \\ 0 & 1 & -2 & -2 \end{array}\right)$$

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The first augmented column deals with the first system and the reduced echelon form above tells us that its solution is $(7, -2)$. The second augmented column relates to the second system and gives the solution $(5, -2)$.

This observation allows us to solve matrix equations: For example, suppose $A$ and $B$ are matrices and we need to solve for a matrix $X$ that satisfies $AX = B$. Then each column of $AX$ must equal the corresponding column of $B$, This observation allows us to solve matrix equations: For example, suppose $A$ and $B$ are matrices and we need to solve for a matrix $X$ that satisfies $AX = B$. Then each column of $AX$ must equal the corresponding column of $B$, that is, $A\mathbf{x}_j = \mathbf{b}_j$.

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$$\text{Solve:} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & -2 \\ 2 & 4 & 8 \end{pmatrix} X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Start with the triply-augmented matrix $(A \mid I)$ and begin applying EROs:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & -2 & 0 & 1 & 0 \\ 2 & 4 & 8 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -5 & -1 & 1 & 0 \\ 0 & 0 & 2 & -2 & 0 & 1 \end{array}\right)$$

Now take $1/2$ times row 3 and continue

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -5 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1/2 \end{array}\right) \xrightarrow[R_1-3R_3]{R_2+5R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & 0 & -3/2 \\ 0 & 1 & 0 & -6 & 1 & 5/2 \\ 0 & 0 & 1 & -1 & 0 & 1/2 \end{array}\right)$$

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Finally,

$$\xrightarrow{R_1-2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 16 & -2 & -13/2 \\ 0 & 1 & 0 & -6 & 1 & 5/2 \\ 0 & 0 & 1 & -1 & 0 & 1/2 \end{array}\right)$$

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This tells us that the columns of $X$ are

$$\begin{pmatrix} 16 \\ -6 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -13/2 \\ 5/2 \\ 1/2 \end{pmatrix} \text{ so } X = \begin{pmatrix} 16 & -2 & -13/2 \\ -6 & 1 & 5/2 \\ -1 & 0 & 1/2 \end{pmatrix}$$

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$$\left( \begin{array}{c|c} A & I \end{array} \right) \xrightarrow{\text{Some EROs}} \left( \begin{array}{c|c} I & A^{-1} \end{array} \right)$$

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If it turns out that you cannot get the identity in the left half (i.e., one of the columns does not have a leading 1), then $A$ is not invertible.

## Definition

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$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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## Theorem

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Example, using $E_3$ from above

$$E_3 \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

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$$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_2^{-1} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3^{-1} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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For theoretical purposes we can always represent the Gauss-Jordan process as a sequence of multiplications by elementary matrices. For example, if three EROs can turn $A$ into the identity, and if $E_1$, $E_2$ and $E_3$ are the corresponding elementary matrices, then $A$ is invertible and

$$E_3(E_2(E_1 A)) = I \;\text{ means }\; (E_3 E_2 E_1)A = I, \;\text{ so }\; A^{-1} = E_3 E_2 E_1$$

From $A^{-1} = E_3 E_2 E_1$ we also get

$$A = (A^{-1})^{-1} = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}$$

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If we let $M = E_m \cdots E_2 E_1$ then this is the same as

$$M \left( \begin{array}{c|c} A & I \end{array} \right) = \left( \begin{array}{c|c} MA & MI \end{array} \right) = \left( \begin{array}{c|c} I & M \end{array} \right)$$

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3. $A\mathbf{x} = \mathbf{b}$ *has a solution for any choice of* $\mathbf{b}$*.*

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*Moreover, in this case $A\mathbf{x} = \mathbf{b}$ always has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.*