Matrices, cont.

D. H. Luecking

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Matrix algebra

If $A=(a_{ij})_{n\times k}$ and $B=(b_{ij})_{k\times m}$, then the entries of AB are

$$\vec{\mathbf{a}}_i\mathbf{b}_j=\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{ik}\end{pmatrix}\begin{pmatrix} b_{1j}\\ b_{2j}\\ \vdots\\ b_{kj}\end{pmatrix}=\sum_{p=1}^{k}a_{ip}b_{pj}$$

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It follows from this that if C is also $k\times m$ then $A(B+C)=AB+AC$.

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Note that we *do not* always have $AB = BA$. Here is a simple example:

If we let $A = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then

$$AB = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{but} \quad BA = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$$

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Definition

If A is a square matrix $(n \times n)$ and if there is another $n \times n$ matrix B that satisfies $AB = I = BA$, then we say A is *invertible* and B is its inverse. We write $B = A^{-1}$.

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For example, if we consider the equation $AB = I$ it would look something like this:

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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If A has an inverse, and we are able to compute it, then we have an advantage in solving systems of equations. Every system can be expressed as a matrix equation $A\mathbf{x} = \mathbf{b}$. If A is invertible, then

$$\mathbf{x} = I\mathbf{x} = (A^{-1}A)\mathbf{x} = A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$$

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Properties of inverses

If A is invertible then so is A^{-1} and the inverse of A^{-1} is A.

If A and B are invertible then so is AB, and the inverse of AB is $B^{-1}A^{-1}$.

To see this we only have to verify that $(AB)(B^{-1}A^{-1}) = I$ and $(B^{-1}A^{-1})(AB) = I$.

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Because of the regrouping property the parentheses are unnecessary in a string of multiplications. That is $(AB)(CD)$ could just as we be written $ABCD$. This simplifies some calculations:

$$ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I.$$

The transpose of a matrix

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Note that the transpose of a row vector is a column vector and vice versa:

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_k \end{pmatrix}^T = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}^T = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \end{pmatrix}$$

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Properties of the transpose

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3. $(A+B)^T = A^T + B^T$
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The transpose allows us to multiply vectors together: If $\mathbf{a}$ and $\mathbf{b}$ are column vectors, we get an inner product from $\mathbf{a}^T\mathbf{b}$ and an outer product from $\mathbf{a}\mathbf{b}^T$.