Gauss-Jordan Reduction

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The system we will address is the following

$$\begin{aligned}
x_1 + 2x_2 + x_3 + x_4 - x_5 &= 1\\
x_1 + 2x_2 + 2x_3 + 3x_4 - x_5 &= 3\\
-2x_1 - 4x_2 + x_4 + x_5 &= 4\\
x_1 + 2x_2 + 2x_3 + 4x_4 &= 1
\end{aligned}$$

The first step is to write out the augmented matrix:

$$\left(\begin{array}{rrrrr|r} 1 & 2 & 1 & 1 & -1 & 1 \\ 1 & 2 & 2 & 3 & -1 & 3 \\ -2 & -4 & 0 & 1 & 1 & 4 \\ 1 & 2 & 2 & 4 & 0 & 1 \end{array}\right)$$

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As we've seen, we want to eliminate x_1 from all equations except the first. In terms of the matrix, this means applying operations on the rows that produce zeros in the first column of rows 2 through 4:

$$\xrightarrow[\substack{R_3+2R_1 \\ R_4-R_1}]{R_2-R_1} \left(\begin{array}{rrrrr|r} 1 & 2 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 2 & 3 & -1 & 6 \\ 0 & 0 & 1 & 3 & 1 & 0 \end{array}\right)$$

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$$\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 2 & 3 & -1 & 6 \\ 0 & 0 & 1 & 3 & 1 & 0 \end{array}\right) \xrightarrow[\substack{R_3-2R_2 \\ R_4-R_2}]{} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{array}\right)$$

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One last step produces a form that is as close as we can get to triangular form.

Using the R_3 as a pivot row, we produce a zero in the 4th column of the fourth row (i.e., we eliminate x_4).

$$\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{array}\right) \xrightarrow[R_4+R_3]{} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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This corresponds to the following system:

$$\begin{aligned} x_1 + 2x_2 + x_3 + \quad x_4 - x_5 &= 1 \\ x_3 + \ 2x_4 \qquad &= 2 \\ -x_4 - x_5 &= 2 \\ 0 &= 0 \end{aligned}$$

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We could apply a process similar to back substitution, but we can make this a little easier if we process the matrix a few more steps.

$$\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[-R_3]{} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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This produces a matrix that is said to be in *row-echelon form*.

$$\left(\begin{array}{rrrrr|r} 1 & 2 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[-R_3]{} \left(\begin{array}{rrrrr|r} 1 & 2 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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This produces a matrix that is said to be in *row-echelon form*. We'll discuss that later. The following several steps produce *reduced row-echelon form*.

$$\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[-R_3]{} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This produces a matrix that is said to be in *row-echelon form*. We'll discuss that later. The following several steps produce *reduced row-echelon form*. We start with the last nonzero row and use it as a pivot to produce zeros above the 1 in column 4:

$$\xrightarrow[R_2-2R_3]{R_1-R_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 0 & -2 & 6 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Then we do the same with the next row above that to get zero above the 1 in column 3:

$$\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 0 & -2 & 6 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 - R_2} \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -2 & 6 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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This corresponds to the following system

$$\left.\begin{aligned} x_1 + 2x_2 \qquad\qquad\qquad\qquad &= -3 \\ x_3 \qquad - 2x_5 &= 6 \\ x_4 + \ x_5 &= -2 \\ 0 &= 0 \end{aligned}\right\} \quad \text{or} \quad \begin{cases} x_1 = -3 - 2x_2 \\ x_3 = 6 + 2x_5 \\ x_4 = -2 - x_5 \end{cases}$$

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If we set $x_2 = \alpha$ and $x_5 = \beta$ we get the following solutions: $(-3 - 2\alpha, \alpha, 6 + 2\beta, -2 - \beta, \beta)$.

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The following matrices are in echelon form:

$$\begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The following matrices are *not* in echelon form

$$\begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 2 & 6 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 2 & 4 \end{pmatrix}$$

The following matrices are in *reduced row-echelon form*:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & -7 & -8 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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2. If we multiply a row by α we can undo that by multiplying the same row by $1/\alpha$.
3. If we change R_k by adding αR_j to it, we can change R_k back by adding $-\alpha R_j$ to it.

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1. If the column is not all zeros make the top of the column nonzero by exchanging it with a row below it if necessary. This top row will then be the pivot row.

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0. If the column is all zeros ignore it, move to the next column and start again.
1. If the column is not all zeros make the top of the column nonzero by exchanging it with a row below it if necessary. This top row will then be the pivot row.
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If we apply EROs to an augmented matrix, the result is equivalent to the starting matrix. The key ideas of this section are

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4. Apply all of the above to the part of the matrix below that row. Repeat until the matrix is in echelon form.

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Step 1 requires us to exchange rows 1 and 2 to get a nonzero entry at the top of column 1. Then step 2 will require us to multiply the new first row by $1/2$:

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cccc|c} 2 & 4 & 6 & 0 & 4 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & 5 & 2 \\ 0 & 3 & -3 & 7 & 4 \end{array}\right) \xrightarrow{(1/2)R_1} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 2 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & 5 & 2 \\ 0 & 3 & -3 & 7 & 4 \end{array}\right)$$

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$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 2 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & 5 & 2 \\ 0 & 3 & -3 & 7 & 4 \end{array}\right) \xrightarrow[R_4-3R_2]{R_3-2R_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 2 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

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Now we ignore rows 1 and 2 and work with the last 2 rows. Again, steps 1 and 2 can be skipped and we apply step 3 once to get a zero in column 4 below the leading 1.

$$\xrightarrow[R_4-R_3]{} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 2 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

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Note also that (I won't write out the details) *if that row/equation were not present* the system would look like the following:

$$\left.\begin{aligned} x_1 + 2x_2 + 3x_3 \qquad\quad &= 2 \\ x_2 - \ \ x_3 + 2x_4 &= 1 \\ x_4 &= 0 \end{aligned}\right\} \quad \text{or} \quad \begin{cases} x_1 = -5x_3 \\ x_2 = 1 + x_3 \\ x_4 = 0 \end{cases}$$

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This leads us the following rules for determining the number of solutions.

1. If the echelon form of the augmented matrix has a row which is zero in the system part and nonzero in the augmented part, then there are no solutions.

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From the rules we developed we see that it has infinitely many solutions (second rule) because the leading 1's in columns 1, 2 and 4 correspond to leading variables x_1, x_2, x_4. This means x_3 is a free variable.

We can perform some more EROs (working from the bottom) to get reduced echelon form:

$$\xrightarrow{R_2-2R_3} \left(\begin{array}{cccc|c} 1 & 3 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1-3R_2} \left(\begin{array}{cccc|c} 1 & 0 & 5 & 0 & 10 \\ 0 & 1 & -1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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$$\left.\begin{array}{rl} x_1 \qquad + 5x_3 \qquad &= \;\; 10 \\ x_2 - \;\; x_3 \qquad &= -3 \\ x_4 &= \;\;\; 2 \end{array}\right\} \text{ or } \left\{\begin{array}{l} x_1 = 10 - 5x_3 \\ x_2 = -3 + x_3 \\ x_4 = 2 \end{array}\right.$$

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Then all solutions arise from setting the nonleading variables (in this case just x_3) to arbitrary values and using these equations to get formulas for the leading variables.

For example: $x_3 = \alpha$ leads to $x_1 = 10 - 5\alpha$, $x_2 = -3 + \alpha$ and $x_4 = 2$.

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