Systems of Equations and Matrices

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$$x_1 + 2x_2 = 6 + 3x_3 \quad \longrightarrow \quad 1x_1 + 2x_2 + (-3)x_3 = 6$$

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The following are not linear

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Also, the nonlinear $(x_1 + x_2)^2 = 4$ is equivalent to two linear equations: $x_1 + x_2 = 2$ or $x_1 + x_2 = -2$. Howvever, we must remember that they cannot both be true at the same time.

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$$\begin{aligned} 2x_1 + 3x_2 - 4x_3 &= 0 \\ 3x_1 + 4x_2 \phantom{{}- 4x_3} &= 2 \end{aligned}$$

The above is referred to as a 2×3 *system*: 2 equations and 3 variables.

A *solution* of an $n \times k$ system with variables x_1 through x_k is a k-tuple of numbers $(s_1, s_2, \ldots, s_k)$ such that the substitutions $x_1 = s_1$ through $x_k = s_k$ make all the equations true.

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We can get an idea as to what is possible for solutions by considering 2×2 systems:

$$\text{S1:} \begin{Bmatrix} 2x_1 - x_2 = 3 \\ 3x_1 + 2x_2 = 1 \end{Bmatrix} \quad \text{S2:} \begin{Bmatrix} 2x_1 - x_2 = 3 \\ 4x_1 - 2x_2 = 6 \end{Bmatrix} \quad \text{S3:} \begin{Bmatrix} 2x_1 - x_2 = 3 \\ 4x_1 - 2x_2 = 0 \end{Bmatrix}$$

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We will see that this pattern holds for any linear system: there is either no solution, 1 solution, or infinitely many solutions.

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$$\begin{aligned} 2x_1 + 4x_2 - 6x_3 &= 2 \\ 3x_1 + 5x_2 - 5x_3 &= 0 \\ -2x_1 - 3x_2 + 3x_3 &= 4 \end{aligned}$$

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3. Add a multiple of one equation to another (replacing the second one). The following is equivalent to the previous:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 1 \\ -x_2 + 4x_3 &= -3 \quad [E_2 - 3E_1] \\ x_2 - 3x_3 &= 6 \quad [E_3 + 2E_1] \end{aligned}$$

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The solution is $(-20, 15, 3)$. Systems in strict triangular form always have exactly one solution. Therefore, not every system can be put in this form.

Consider the three 2×2 systems from before:

$$\text{S1: } \begin{Bmatrix} 2x_1 - x_2 = 3 \\ 3x_1 + 2x_2 = 1 \end{Bmatrix} \quad \text{S2: } \begin{Bmatrix} 2x_1 - x_2 = 3 \\ 4x_1 - 2x_2 = 6 \end{Bmatrix} \quad \text{S3: } \begin{Bmatrix} 2x_1 - x_2 = 3 \\ 4x_1 - 2x_2 = 0 \end{Bmatrix}$$

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Following nearly the same process from before (take $1/2$ the first equation and then adding appropriate multiples of that to the second) we get:

$$\begin{Bmatrix} x_1 - \frac{1}{2}x_2 = \frac{3}{2} \\ \frac{7}{2}x_2 = -\frac{7}{2} \end{Bmatrix} \qquad \begin{Bmatrix} x_1 - \frac{1}{2}x_2 = \frac{3}{2} \\ 0x_2 = 0 \end{Bmatrix} \qquad \begin{Bmatrix} x_1 - \frac{1}{2}x_2 = \frac{3}{2} \\ 0x_2 = -6 \end{Bmatrix}$$

The back substitution method on S1 gives $x_2 = -1$ and then $x_1 - (1/2)(-1) = 3/2$ gives $x_1 = 1$ for the solution $(1, -1)$.

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Still, the method works in that it allows us to reach these conclusions.

Matrices associated with a system

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We have a *system matrix*, which consists of all the numbers that are multiplied by the variables, and an *augmented matrix*, which is the system matrix augmented with the numbers on the right side of the equations. For the system

$$\begin{aligned} 2x_1 + 4x_2 - 6x_3 &= 2 \\ 3x_1 + 5x_2 - 5x_3 &= 0 \\ -2x_1 - 3x_2 + 3x_3 &= 4 \end{aligned}$$

the first matrix below is the augmented matrix and the second is the system matrix:

$$A = \left(\begin{array}{rrr|r} 2 & 4 & -6 & 2 \\ 3 & 5 & -5 & 0 \\ -2 & -3 & 3 & 4 \end{array} \right), \quad B = \left(\begin{array}{rrr} 2 & 4 & -6 \\ 3 & 5 & -5 \\ -2 & -3 & 3 \end{array} \right)$$

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The elimination method we used on the system really only needs to keep track of the numbers and their positions (i.e., which variables they are attached to). As such, we don't need to operate on the system: we can do everything with just the augmented matrix. Instead of manipulating *equations* we manipulate *rows*.

Each row corresponds to one of the equations and each column corresponds to one of the variables. If a system is $n \times k$ then the system matrix has n rows and k columns and we call it an $n \times k$ matrix. The augmented matrix has an extra column and so is $n \times (k + 1)$. The vertical line in the augmented matrix is optional, but helps with keeping track of things.

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3. Add a multiple of one row to another (replacing the second one). The following is equivalent to the previous:

$$\xrightarrow[R_3+2R_1]{R_2-3R_1} \left(\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & -1 & 4 & -3 \\ 0 & 1 & -3 & 6 \end{array}\right)$$

If we proceed one more step, replacing R_3 with $R_2 + R_3$ we get the augmented matrix of a strictly triangular system:

$$\left(\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & -1 & 4 & -3 \\ 0 & 1 & -3 & 6 \end{array}\right) \xrightarrow[R_3+R_2]{} \left(\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & -1 & 4 & -3 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

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Note: Up to here we worked downward. It is useful to now work upwards. In fact, the following additional operations produce something called a *diagonal system*:

$$\xrightarrow[R_2-4R_3]{R_1+3R_3} \left(\begin{array}{rrr|r} 1 & 2 & 0 & 10 \\ 0 & -1 & 0 & -15 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{R_1+2R_2} \left(\begin{array}{rrr|r} 1 & 0 & 0 & -20 \\ 0 & -1 & 0 & -15 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

This last one produces the easy-to-solve system $x_1 = -20$, $-x_2 = -15$, $x_3 = 3$ for the solution $(-20, 15, 3)$.